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Recovering Sturm–Liouville Operators on a Graph from Pairwise Disjoint Spectra

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Research Article

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Abstract

In this paper, we study an inverse spectral problem for the Sturm-Liouville equation on a three-star graph with the Neumann and Dirichlet boundary conditions in the boundary vertices and matching conditions in the internal vertex. As spectral characteristics, we consider the spectrum of the main problem together with the spectra of two Neumann-Dirichlet problems and one Dirichlet-Dirichlet problem on the edges of the graph and investigate their properties and asymptotic behavior. We prove that if these four spectra do not intersect, then the inverse problem of recovering the potential is uniquely solvable . We give an algorithm for the construction of the potential corresponding to this quadruple of spectra.

Keywords: Sturm–Liouville operators, three-star graph,Neumann and Dirichlet boundary conditions, Hermite-Biehler function, inverse spectral problem

2010 Mathematics Subject Classification: 34A55; 34B24; 34B45; 34L05

1 Introduction

This paper is devoted to the study of the inverse spectral problem for Sturm-Liouville operators on a three-star graph with the Neumann and Dirichlet boundary conditions in the boundary vertices and matching conditions in the internal vertex. The inverse problem consists of recovering the potential on a graph from the given spectral characteristics. Differential operators on graphs(networks, trees) often appear in mathematics, mechanics, physics, geophysics, physical chemistry, electronics, nanoscale technology and branches of natural sciences and engineering(see (2; 5; 6; 11; 12; 13; 22; 33) and the bibliographies thereof). In recent years there has been considerable interest in the spectral theory of Sturm-Liouville operators on graphs(see (1; 31; 32)). The direct spectral and scattering problems on compact and noncompact graphs, respectively, were considered in many publications(see, for example (4; 9; 20)). Inverse spectral problems of recovering differential operators on arbitrary trees(i.e., graphs without cycles) and specially star-type graphs with the boundary conditions or

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spectral characteristics other than considered here, were studied in (8; 21; 25; 35; 36) and other papers. Hochstadt-Liberman type inverse problems on star-type graphs were invesigated in (25; 34).

We consider a three-star graph G with vertex set $V = \{v_0, v_1, v_2, v_3\}$ and edge set $E = \{e_1, e_2, e_3\}$, where v_1, v_2, v_3 are the boundary vertices, v_0 is the internal vertex and $e_j = [v_j, v_0]$ for $j = 1, 2, 3$. We assume that the length of every edge is equal to a, $a > 0$. Every edge $e_j \in E$ is viewed as an interval $[0, a]$. Parametrize $e_j \in E$ by $x \in [0, a]$, the following choice of orientation is convenient for us: $x = 0$ corresponds to the boundary vertices v_1, v_2, v_3 and $x = a$ corresponds to the internal vertex v_0. A function Y on G may be represented as a vector $Y(x) = [y_j(x)]_{j=1,2,3}$, $x \in [0, a]$ and the function $y_j(x)$ is defined on the edge e_j. Let $q(x) = [q_j(x)]_{j=1,2,3}$ be a function on G which is called the potential and $q_j(x) \in L_2(0, a)$ is a real-valued function defined on the edge e_j . Let us consider the following Sturm-Liouville equations on G:

$$-y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), \quad x \in [0, a], \quad j = 1, 2, 3, \tag{1.1}$$

where λ is the spectral parameter. The functions $y_j(x)$ and $y_j'(x)$ are absolutely continuous and satisfy the following matching conditions in the internal vertex v_0:

$$\left.\begin{array}{l} y_j(a) = y_{j'}(a) \quad \text{for } j, j' = 1, 2, 3, \quad \text{(continuity condition)}, \\ \displaystyle\sum_{j=1}^{3} y_j'(a) = 0 \quad \text{(Kirchhoff's condition)}. \end{array}\right\} \tag{1.2}$$

In electrical circuits, (1.2) expresses Kirchhof's law; in an elastic string network, it expresses the balance of tension and so on. Let us denote by L_0 the boundary-value problem for (1.1) with the matching conditions (1.2) and the following boundary conditions at the boundary vertices v_1, v_2, v_3:

$$y_1'(0) = y_2'(0) = y_3(0) = 0. \tag{1.3}$$

The problem of small transverse vibrations of a three-star graph consisting of three inhomogeneous smooth strings whose two free ends can move without friction in the directions orthogonal to their respective equilibrium positions and one fixed end can be reduced to this problem by the Liouvile transformation. This problem occurs also in quantum mechanics when one considers a quantum particle subject to the Shrödinger equation moving in a quasi-one-dimensional graph domain.

In this paper, we study the inverse problem of recovering the potential $q(x) = [q_j(x)]_{j=1,2,3}$ from the spectral characteristics. Similar inverse spectral problems on star-type graphs with three and arbitrary number of edges but only with the Dirichlet conditions at the boundary vertices was considered in (25; 27). Our method follows (25; 27). As spectral characteristics, we consider the set of eigenvalues of problem L_0 together with the sets of eigenvalues of the following two Neumann - Direchlet problems and one Dirichlet- Dirichlet problem on the edges of the graph G:

$$\left\{\begin{array}{l} -y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), \quad x \in [0, a], \\ y_j^{(n_j)}(0) = y_j(a) = 0, \quad j = 1, 2, 3, \quad n_1 = n_2 = 1, n_3 = 0, \end{array}\right. \tag{1.4}$$

which we denote these problems by L_j, $j = 1, 2, 3$. We obtain conditions for four sequences of real numbers that enable one to reconstruct the potential $q(x) = [q_j(x)]_{j=1,2,3}$ so that one of the sequences describes the spectrum of the boundary-value problem L_0 and other three sequences coincide with the spectra of the problems L_j, $j = 1, 2, 3$. We give an algorithm for the construction of the potential from these four sequences.

The main idea of the solution of the inverse problem for the considered system is its reduction to three independent inverse problems of reconstruction of the functions $q_j(x) \in L_2(0, a)$, $j = 1, 2, 3$, on the basis of two spectra, namely, the spectrum of the problem L_j and the spectrum of the following boundary-value problem on the edge e_j:

$$\left\{\begin{array}{l} -y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), \quad x \in [0, a], \\ y_j^{(n_j)}(0) = y_j'(a) = 0, \quad j = 1, 2, 3, \quad n_1 = n_2 = 1, n_3 = 0. \end{array}\right. \tag{1.5}$$

Let us denote this problem by $L'_j (j = 1, 2, 3)$. Since the solutions of the later inverse problems are known(see (18, Sec.3.4), (7, Sec.1.5)), this reduction gives an algorithm for the reconstruction of the potential of the boundary-value problem L_0.

Let us consider the operator-theoretical interpretation of our problem. Denote by A the operator acting in the Hilbert space $H = L_2(0, a) \oplus L_2(0, a) \oplus L_2(0, a)$ with standard inner product $(.,.)_H$, according to the formulas

$$AY = A \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} = \begin{pmatrix} -y''_1(x) + q_1(x)y_1(x) \\ -y''_2(x) + q_2(x)y_2(x) \\ -y''_3(x) + q_3(x)y_3(x) \end{pmatrix}, \tag{1.6}$$

$$D(A) = \left\{ \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{pmatrix} \middle| \begin{array}{l} y_j(x) \in W_2^2(0, a) \quad \text{for } j = 1, 2, 3, \\ y_j(a) = y_{j'}(a) \quad \text{for } j, j' = 1, 2, 3, \\ \sum_{j=1}^{3} y'_j(a) = 0, \quad y'_1(0) = y'_2(0) = y_3(0) = 0 \end{array} \right\}, \tag{1.7}$$

where $W_2^2(0, a)$ is a Sobolev space. Let us show that $D(A)$ is dense in H. Suppose that $F = (f_1(x), f_2(x), f_3(x))^t \in H$ is orthogonal to all $G = (g_1(x), g_2(x), g_3(x))^t \in D(A)$($t$ denotes the transpose of a matrix), i.e.,

$$\langle F, G\rangle_H = \sum_{j=1}^{3} \int_0^a f_j(x)g_j(x) = 0.$$

Since $C_0^\infty[0, a] \oplus 0 \oplus 0 \subseteq D(A)$(Here 0 is a function that identically zero on $[0, a]$), then $G = (g_1(x), 0, 0) \in C_0^\infty[0, a] \oplus 0 \oplus 0$ is orthogonal to F, i.e.,

$$\langle F, G\rangle_H = \int_0^a f_1(x)g_1(x) = 0.$$

Since $C_0^\infty[0, a]$ is dense in $L_2(0, a)$, we must have $f_1(x) = 0$. Similarly, we get that $f_2(x) = f_3(x) = 0$. Thus, D(A) is dense in H. We prove that A is self-adjoint in the Hilbert space H. Let $F = (f_1(x), f_2(x), f_3(x))^t$ and $G = (g_1(x), g_2(x), g_3(x))^t$ be arbitrary elements of $D(A)$. By twice integration by parts, we have

$$\langle AF, G\rangle_H = \langle F, AG\rangle_H + \sum_{j=1}^{3} \left(f_j g'_j - f'_j g_j\right)\Big|_0^a .$$

It follows from (1.2) and (1.3) that $\sum_{j=1}^{3} \left(f_j g'_j - f'_j g_j\right)\Big|_0^a = 0$. This yields,

$$\langle AF, G\rangle_H = \langle F, AG\rangle_H.$$

Therefore, A is symmetric in H. It remains to show that if $(AY, V)_H = (Y, U)_H$ for all $Y = (y_1(x), y_2(x), y_3(x))^t \in D(A)$, then $V \in D(A)$ and $AV = U$, where $V = (v_1(x), v_2(x), v_3(x))^t$ and $U = (u_1(x), u_2(x), u_3(x))^t$, i.e., (i) $v_j(x) \in W_2^2(0, a)\,(j = 1, 2, 3)$; (ii) $v'_1(0) = v'_2(0) = v_3(0) = 0$; (iii) $v_j(a) = v_{j'}(a)\ \ (j, j' = 1, 2, 3)$; (iv) $\sum_{j=1}^{3} v'_j(a) = 0$; (v) $\ell_j y_j = u_j\,(j = 1, 2, 3)$, where $\ell_j y_j := -y''_j + q_j y_j$.

For all $Y \in C_0^\infty(0, a) \oplus 0 \oplus 0 \subseteq D(A)$($0$ denotes the function identically zero on $[0, a]$), we have

$$\int_0^a (\ell_1 y_1)\overline{v}_1 dx = \int_0^a y_1 \overline{u}_1 dx.$$

So by standard Sturm–Liouville theory $v_1(x) \in W_2^2(0, a)$ and $u_1 = \ell_1 v_1$. Similarly we get $v_j(x) \in W_2^2(0, a)$ and $u_j = \ell_j v_j\,(j = 2, 3)$. Thus (i) and (v) hold. Now using (v) equation $(AY, V)_H = (Y, U)_H$ for all $Y \in D(A)$ becomes

$$\sum_{j=1}^{3} \int_0^a (\ell_j y_j)\overline{v}_j dx = \sum_{j=1}^{3} \int_0^a y_j \overline{\ell_j v_j} dx.$$

However by twice integration by parts, we have

$$\sum_{j=1}^{3}\int_0^a (\ell_j y_j)\overline{v}_j dx = \sum_{j=1}^{3}\int_0^a y_j\overline{\ell_j v_j}dx + \sum_{j=1}^{3}\left(y_j\overline{v}_j' - y_j'\overline{v}_j\right)\Big|_0^a.$$

Hence

$$\sum_{j=1}^{3}\left(y_j\overline{v}_j' - y_j'\overline{v}_j\right)\Big|_0^a = 0. \tag{1.8}$$

According to Naimark's patching lemma(see (23, p. 63, Lemma 2)), there exists a $Y \in D(A)$ such that $y_1(0) = 1,\ y_2(0) = y_3'(0) = y_1(a) = y_1'(a) = y_2(a) = y_2'(a) = y_3(a) = y_3'(a) = 0$. Then on account of equality (1.8), we have $v_1'(0) = 0$. Similarly, we get $v_2'(0) = v_3(0) = 0$. So (ii) holds. Using Naimark's patching lemma again one can show that (iii) and (iv) hold. consequently the operator A is self-adjoint.

The operator A has a discrete spectrum and its eigenvalues coincide with the squares of the eigenvalues of the boundary-value problem L_0. Furthermore, integrating by parts, we obtain the following equality for any vector function $Y = (y_1(x), y_2(x), y_3(x))^t \in D(A)$:

$$(AY, Y)_H = \sum_{j=1}^{3}\int_0^a (|y_j'(x)|^2 + q_j(x)|y_j(x)|^2)dx. \tag{1.9}$$

Thus, for all eigenvalues of the boundary-value problem L_0 to be real and nonzero, it is necessary and sufficient that the operator A be strictly positive($A \gg 0$). Relation (1.9) yields the following simple sufficient condition for the strict positivity of the operator A:

$$q_j(x) \geq 0 \quad \text{a.e. on } [0, a], \quad j = 1, 2, 3. \tag{1.10}$$

On the other hand, if $A \gg 0$, then all eigenvalues of the operator A are real and positive, otherwise the strict positivity of the operator A can be realized by shifting the spectral parameter $\lambda^2 - q_0,\ q_0 > 0$, in (1.1). For this reason, we assume in what follows without loss of generality that the condition (1.10) is valid. Thus the eigenvalues of the boundary-value problems L_0 and L_j for $j = 1, 2, 3$ are nonzero real numbers.

This paper has the following structure: In section 2 the direct problem is considered. Aspects of the theory of operator pencils in combination with methods of the theory of entire functions are used as tools for a description of the set of eigenvalues of the boundary-value problem L_0 and the spectra of the auxiliary problems L_j for $j = 1, 2, 3$ associated with this system. As a consequence we prove that the eigenvalues of the main problem and the spectra of the auxiliary problems interlace in some sense. In section 3 we solve the inverse spectral problem for L_0 within the framework of the statement indicated above.

2 Direct problem

In this section, we describe the properties of sequences of eigenvalues of the boundary-value problems L_0 and L_j for $j = 1, 2, 3$ that are necessary for what follows.

Let us denote by $c_j(x, \lambda)$, $s_j(x, \lambda)$ the solutions of (1.1) on the edge e_j for $j = 1, 2, 3$ which satisfy the initial conditions

$$c_j'(0, \lambda) = c_j(0, \lambda) - 1 = 0, \qquad s_j(0, \lambda) = s_j'(0, \lambda) - 1 = 0, \quad j = 1, 2, 3. \tag{2.1}$$

For each fixed $x \in [0, a]$, the functions $c_j^{(\nu)}(x, \lambda)$ and $s_j^{(\nu)}(x, \lambda)$, $\nu = 0, 1,\ j = 1, 2, 3$ are entire in λ. Since $\{c_j(x, \lambda), s_j(x, \lambda)\}$ is a fundamental system of solutions of (1.1) on the edge e_j, then the solutions of (1.1) for $j = 1, 2, 3$ that satisfy the conditions (1.3), are

$$y_j(x, \lambda) = \begin{cases} C_j c_j(x, \lambda), & j = 1, 2, \\ C_3 s_3(x, \lambda), & j = 3, \end{cases} \tag{2.2}$$

where C_j, $j = 1, 2, 3$ are constants. Substituting (2.2) into (1.2), we establish that the eigenvalues of the boundary-value problem L_0 are zeros of the entire function

$$\varphi_1(\lambda) := \begin{vmatrix} c_1(a,\lambda) & -c_2(a,\lambda) & 0 \\ c_1(a,\lambda) & 0 & -s_3(a,\lambda) \\ c_1'(a,\lambda) & c_2'(a,\lambda) & s_3'(a,\lambda) \end{vmatrix}$$

or

$$\varphi_1(\lambda) = c_1(a,\lambda)c_2'(a,\lambda)s_3(a,\lambda) + c_1'(a,\lambda)c_2(a,\lambda)s_3(a,\lambda) + c_1(a,\lambda)c_2(a,\lambda)s_3'(a,\lambda). \tag{2.3}$$

Let us represent $\varphi_1(\lambda)$ by three equivalent formulas

$$\varphi_1(\lambda) = \psi_{j1}(\lambda)u_{j1}(\lambda) + \psi_{j2}(\lambda)u_{j2}(\lambda), \quad j = 1, 2, 3, \tag{2.4}$$

where

$$\begin{pmatrix} \psi_{11}(\lambda) & \psi_{12}(\lambda) \\ \psi_{21}(\lambda) & \psi_{22}(\lambda) \\ \psi_{31}(\lambda) & \psi_{32}(\lambda) \end{pmatrix} = \begin{pmatrix} c_1(a,\lambda)c_2'(a,\lambda) + c_1'(a,\lambda)c_2(a,\lambda) & c_1(a,\lambda)c_2(a,\lambda) \\ c_1(a,\lambda)s_3'(a,\lambda) + c_1'(a,\lambda)s_3(a,\lambda) & c_1(a,\lambda)s_3(a,\lambda) \\ c_2(a,\lambda)s_3'(a,\lambda) + c_2'(a,\lambda)s_3(a,\lambda) & c_2(a,\lambda)s_3(a,\lambda) \end{pmatrix}, \tag{2.5}$$

$$\begin{pmatrix} u_{11}(\lambda) & u_{12}(\lambda) \\ u_{21}(\lambda) & u_{22}(\lambda) \\ u_{31}(\lambda) & u_{32}(\lambda) \end{pmatrix} = \begin{pmatrix} s_3(a,\lambda) & s_3'(a,\lambda) \\ c_2(a,\lambda) & c_2'(a,\lambda) \\ c_1(a,\lambda) & c_1'(a,\lambda) \end{pmatrix}. \tag{2.6}$$

Lemma 2.1. *All zeros of the functions $\psi_{j1}(\lambda)$, $j = 1, 2, 3$ are simple.*

Proof. The zeros of the function $\psi_{11}(\lambda)$ coincides with the spectrum of the problem

$$\begin{cases} -y_j''(x) + q_j(x)y_j(x) = \lambda^2 y_j(x), \quad x \in [0,a], \quad j = 1, 2, \\ y_1'(0) = y_2'(0) = 0, \\ y_1(a) = y_2(a), \\ y_1'(a) + y_2'(a) = 0 \end{cases}$$

or what is the same, with the spectrum of the problem

$$-\widetilde{y}''(x) + \widetilde{q}(x)\widetilde{y}(x) = \lambda^2\widetilde{y}(x), \quad x \in [0, 2a], \tag{2.7}$$

$$\widetilde{y}'(0) = \widetilde{y}'(2a) = 0, \tag{2.8}$$

where

$$\widetilde{y}(x) = \begin{cases} y_1(x) & \text{if } x \in [0,a], \\ y_2(2a - x) & \text{if } x \in (a, 2a], \end{cases}$$

$$\widetilde{q}(x) = \begin{cases} q_1(x) & \text{if } x \in [0,a], \\ q_2(2a - x) & \text{if } x \in (a, 2a]. \end{cases}$$

Due to (1.10), the spectrum of the Neumann problem (2.7), (2.8) are real, nonzero and simple. Therefore, the zeros of $\psi_{11}(\lambda)$ are simple. In the same way, we can show that the zeros of $\psi_{21}(\lambda)$ and $\psi_{31}(\lambda)$ are simple too. The assertion of Lemma 2.1 follows. □

For what follows, we need the definition presented below:

Definition 2.1. ((26)) Let $\{z_k\}_{-\infty}^{\infty}(\{z_k\}_{-\infty,k\neq 0}^{\infty})$ be a sequence of complex numbers of finite multiplicities which satisfy the following conditions: (1) the sequence is symmetric with respect to the imaginary axis and symmetrically located numbers possess the same multiplicities; (2) any strip $|\mathrm{Re}\, z| \leq p < \infty$ contains not more than a finite number of z_k. Then, the following way of enumeration is called proper:

i. $z_{-k} = -\overline{z_k}(\mathrm{Re}\, z_k \neq 0)$;

ii. Re $z_k \leq$ Re z_{k+1};

iii. the multiplicities are taken into account.

If a sequence has even number of pure imaginary elements we exclude the index zero from enumeration to make it proper.

Throughout section 2, denote

$$B_j = \frac{1}{2}\int_0^a q_j(x)dx, \quad j = 1, 2, 3.$$

We introduce the entire function

$$\varphi_2(\lambda) = c_1(a, \lambda)c_2(a, \lambda)s_3(a, \lambda). \tag{2.9}$$

Let us denote by $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ the set of zeros of $\varphi_1(\lambda)$ and by $\{\theta_k\}_{-\infty,k\neq 0}^{\infty}$ the set of zeros of the function $\varphi_2(\lambda)$. Denote by $\{\nu_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$, $\{\nu_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\nu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ the sets of zeros of the functions $c_1(a, \lambda)$, $c_2(a, \lambda)$ and $s_3(a, \lambda)$, respectively. It is clear from (2.9) that the set $\{\theta_k\}_{-\infty,k\neq 0}^{\infty}$ is the union of the sets $\bigcup_{j=1}^{3}\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, i.e., the spectra of the auxiliary problems L_j for $j = 1, 2, 3$. According to the remark presented in section 1, all numbers τ_k, $\nu_k^{(j)}$, $j = 1, 2, 3$ and θ_k are real and nonzero. We enumerate the sets $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$, $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, $j = 1, 2, 3$ and $\{\theta_k\}_{-\infty,k\neq 0}^{\infty}$ in the proper way($\tau_{-k} = -\tau_k$, $\tau_k \leq \tau_{k+1}$, $\nu_{-k}^{(j)} = -\nu_k^{(j)}$, $\nu_k^{(j)} < \nu_{k+1}^{(j)}$, $j = 1, 2, 3$ and $\theta_{-k} = -\theta_k$, $\theta_k \leq \theta_{k+1}$). Note that the sets of eigenvalues $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, $j = 1, 2, 3$ behave asypmtotically as follows(see (18, section 1.5)):

$$\nu_k^{(j)} = \frac{\pi\left(k - \frac{1}{2}\right)}{a} + \frac{B_j}{\pi\left(k - \frac{1}{2}\right)} + \frac{\delta_k^{(j)}}{k}, \quad j = 1, 2, \tag{2.10}$$

$$\nu_k^{(3)} = \frac{\pi k}{a} + \frac{B_3}{\pi k} + \frac{\delta_k^{(3)}}{k}, \tag{2.11}$$

where $\{\delta_k^{(j)}\}_{-\infty k\neq 0}^{\infty} \in l_2$ for $j = 1, 2, 3$.

Lemma 2.2. *1. If $\tau_k = \theta_n$ for some k and n, then $\frac{d\varphi_2(\lambda)}{d\lambda} = 0$, i.e., at least two of three functions $c_1(a, \lambda)$, $c_2(a, \lambda)$ and $s_3(a, \lambda)$ have (simple)zeros at $\lambda = \tau_k = \theta_n$.*
2. If $\tau_k = \theta_n$ for some k and n and $\left.\frac{d\varphi_1(\lambda)}{d\lambda}\right|_{\lambda=\theta_n} = 0$, then $c_1(a, \lambda) = c_2(a, \lambda) = s_3(a, \lambda) = 0$ and $\left.\frac{d^2\varphi_1(\lambda)}{d\lambda^2}\right|_{\lambda=\theta_n} \neq 0$.

Proof. Since θ_n is a zero of $\varphi_2(\lambda)$, hence, definition (2.9) implies that at least one of three functions $c_1(a, \lambda)$, $c_2(a, \lambda)$ and $s_3(a, \lambda)$ has (simple) zeros at $\lambda = \tau_k = \theta_n$, say, $s_3(a, \theta_n) = 0$. Then from $\varphi_1(\theta_n) = \varphi_1(\tau_k)$ and (2.3) we obtain $c_1(a, \theta_n)c_2(a, \theta_n) = 0$ and assertion 1 of Lemma 2.2 follows. If we assume $c_2(a, \lambda) = 0$ or $c_2(a, \lambda) = 0$, then the proof is analogous.

If $\tau_k = \theta_n$ for some k and n, then by assertion 1 at least two of the functions $c_1(a, \lambda)$, $c_2(a, \lambda)$ and $s_3(a, \lambda)$ have (simple)zeros at $\lambda = \theta_n$. Let $c_1(a, \theta_n) = c_2(a, \theta_n) = 0$. Now let $\left.\frac{d\varphi_1(\lambda)}{d\lambda}\right|_{\lambda=\theta_n} = 0$; then from (2.4) for $j = 1$ we get

$$\begin{aligned}\left.\frac{d\varphi_1(\lambda)}{d\lambda}\right|_{\lambda=\theta_n} &= s_3(a, \theta_n)\left.\frac{d\psi_{11}(\lambda)}{d\lambda}\right|_{\lambda=\theta_n} \\ &= 0.\end{aligned} \tag{2.12}$$

By Lemma 2.1, $\left.\frac{d\psi_{11}(\lambda)}{d\lambda}\right|_{\lambda=\theta_n} \neq 0$ and hence (2.12) implies that $s_3(a,\theta_n) = 0$. If we assume that $c_1(a,\theta_n) = s_3(a,\theta_n) = 0$ or $c_2(a,\theta_n) = s_3(a,\theta_n) = 0$ then the proof of first part of assertion 2 is analogous. Differentiating (2.4) twice for $j = 1$ we calculate

$$\left.\frac{d^2\varphi_1(\lambda)}{d\lambda^2}\right|_{\lambda=\theta_n} = 2\left.\frac{d\psi_{11}(\lambda)}{d\lambda}\right|_{\lambda=\theta_n} \left.\frac{ds_3(a,\lambda)}{d\lambda}\right|_{\lambda=\theta_n}.$$

Since the zeros of $\psi_{11(\lambda)}$ and $s_3(a,\lambda)$ are simple, then $\left.\frac{d\psi_{11}(\lambda)}{d\lambda}\right|_{\lambda=\theta_n} \neq 0$ and $\left.\frac{ds_3(a,\lambda)}{d\lambda}\right|_{\lambda=\theta_n} \neq 0$. Consequently, $\left.\frac{d^2\varphi_1(\lambda)}{d\lambda^2}\right|_{\lambda=\theta_n} \neq 0$. □

Let us denote by L^d, $d > 0$ the class(introduced in (15, p. 149)) of entire functions of exponential type $\leq d$ whose restrictions on the real line belong to $L_2(-\infty,\infty)$.

Let us introduce the function

$$\Xi(\lambda) = \varphi_1(\lambda) + i\alpha\lambda\varphi_2(\lambda), \tag{2.13}$$

where $\alpha \in (3,\infty)$ is an arbitrary constant.

Lemma 2.3. *The function $\Xi(\lambda)$ can be represented as follows:*

$$\begin{aligned}\Xi(\lambda) \;=\;& 3\cos^3\lambda a - 2\cos\lambda a + (2B_1+2B_2+3B_3)\frac{\cos^2\lambda a\sin\lambda a}{\lambda} - (B_1+B_2)\frac{\sin^3\lambda a}{\lambda}\\ &+ i\alpha\left(\cos^2\lambda a\sin\lambda a + (B_1+B_2)\frac{\sin^2\lambda a\cos\lambda a}{\lambda} - B_3\frac{\cos^3\lambda a}{\lambda}\right) + \frac{\omega(\lambda)}{\lambda},\end{aligned} \tag{2.14}$$

where $\omega(\lambda) \in L^{3a}$.

Proof. Using the formulas of (18, p. 9) and taking into account that $\int_0^a f(t)\cos\lambda t dt \in L^a$, $\int_0^a f(t)\sin\lambda t dt \in L^a$ whenever $f \in L_2(0,a)$ by the Paley-Wiener theorem (3, p. 103), we obtain

$$c_j(a,\lambda) \;=\; \cos\lambda a + B_j\frac{\sin\lambda a}{\lambda} + \frac{\upsilon_j(\lambda)}{\lambda,} \quad j = 1,2, \tag{2.15}$$

$$c_j'(a,\lambda) \;=\; -\lambda\sin\lambda a + B_j\cos\lambda a + \varrho_j(\lambda), \quad j = 1,2, \tag{2.16}$$

$$s_3(a,\lambda) \;=\; \frac{\sin\lambda a}{\lambda} - B_3\frac{\cos\lambda a}{\lambda^2} + \frac{\upsilon_3(\lambda)}{\lambda^2}, \tag{2.17}$$

$$s_3'(a,\lambda) \;=\; \cos\lambda a + B_3\frac{\sin\lambda a}{\lambda} + \frac{\varrho_3(\lambda)}{\lambda}, \tag{2.18}$$

where $\upsilon_j(\lambda)$, $\varrho_j(\lambda)$, $j = 1,2,3$, are entire functions of class L^a. Substituting (2.15)-(2.18) into (2.3) and (2.9), we get

$$\begin{aligned}\varphi_1(\lambda) \;=\;& 3\cos^3\lambda a - 2\cos\lambda a + (2B_1+2B_2+3B_3)\frac{\cos^2\lambda a\sin\lambda a}{\lambda}\\ &-(B_1+B_2)\frac{\sin^3\lambda a}{\lambda} + \frac{\omega_1(\lambda)}{\lambda},\end{aligned} \tag{2.19}$$

$$\varphi_2(\lambda) = \frac{\cos^2\lambda a\sin\lambda a}{\lambda} + (B_1+B_2)\frac{\sin^2\lambda a\cos\lambda a}{\lambda^2} - B_3\frac{\cos^3\lambda a}{\lambda^2} + \frac{\omega_2(\lambda)}{\lambda^2}, \tag{2.20}$$

where $\omega_1(\lambda),\omega_2(\lambda) \in L^{3a}$. If we substitute (2.19) and (2.20) into (2.13), then the representation (2.14) follows. □

We set

$$\Xi_0(\lambda) := 3\cos^3\lambda a - 2\cos\lambda a + i\alpha\cos^2\lambda a\sin\lambda a. \tag{2.21}$$

Denote by $\{\lambda_k\}_{-\infty}^{\infty}$ the set of zeros of $\Xi(\lambda)$ and by $\{\lambda_k^{(0)}\}_{-\infty}^{\infty}$ that of $\Xi_0(\lambda)$. Under proper enumeration, it is possible to arrange the zeros $\{\lambda_k^{(0)}\}_{-\infty}^{\infty}$ into two subsequences:

$$\lambda_{2k}^{(0)} = \frac{\pi k}{a} - \frac{i}{a}\log\frac{(-1)^k + \sqrt{\alpha^2 - 8}}{\alpha + 3}, \quad k \in \mathbb{N}\cup\{0\}, \tag{2.22}$$

$$\lambda_{2k-1}^{(0)} = \frac{\pi\left(k - \frac{1}{2}\right)}{a}, \quad k \in \mathbb{N}, \tag{2.23}$$

$$\lambda_{-k}^{(0)} = -\lambda_k^{(0)}, \quad k \in \mathbb{N}.$$

Lemma 2.4. *The zeros of $\Xi(\lambda)$ can be enumerated in such a way that:*

$$\lambda_k = \lambda_k^{(0)} + o(1). \tag{2.24}$$

Proof. The main term of the asymptotic form of $\Xi(\lambda)$ is determined by the term (2.21). Therefore, we consider λ_k's as perturbations of $\lambda_k^{(0)}$'s. We show that the sequence $\{\text{Im}\,\lambda_k\}_{-\infty}^{\infty}$ is bounded. Suppose that there exists a subsequence $\{\lambda_{m_k}\}_{k=-\infty}^{\infty}$ of the sequence $\{\lambda_k\}_{-\infty}^{\infty}$ such that $\text{Im}\,\lambda_{m_k} \to \infty$, as $k \to \infty$. Then (2.14) implies

$$\Xi(\lambda_{m_k}) - e^{-3i\lambda_{m_k}a}\left(\frac{3+\alpha}{8}\right) = o\left(e^{3|\text{Im}\,\lambda_{m_k}|a}\right).$$

Since $\Xi(\lambda_{m_k}) = 0$, this is a contradiction. Hence $\{\text{Im}\,\lambda_k\}_{-\infty}^{\infty}$ is bounded above. In the same way, one can show that $\{\text{Im}\,\lambda_k\}_{-\infty}^{\infty}$ is bounded below. Consequently, there exists a constant $M > 0$ such that $|\text{Im}\,\lambda_k| < M$.
Denote $\Pi = \{\lambda : |\text{Im}\,\lambda| < M + \epsilon\}$, where ϵ is an arbitrary positive number. It follows from (2.14) and (2.21) that there exists a constant $C > 0$ such that

$$|\Xi(\lambda) - \Xi_0(\lambda)| < \frac{C}{|\lambda|}, \quad \lambda \in \Pi.$$

Since the function $\Xi_0(\lambda)$ is periodic, by a method similar to that in (7, p. 6), for every $r \in (0, \epsilon)$, we can find $d > 0$ such that

$$|\Xi_0(\lambda)| > d$$

for all $\lambda \in \Pi \setminus \bigcup_k C_k^{(0)}$, where $C_k^{(0)} = \{\lambda : |\lambda - \lambda_k^{(0)}| \leq r\}$. Taking r sufficiently small we obtain $C_k^{(0)} \bigcap C_{k'}^{(0)} = \emptyset$, $k \neq k'$. Consequently, for all $\lambda \in \{\lambda : \lambda \in \Pi \setminus \bigcup_k C_k^{(0)}, |\lambda| > \frac{C}{d}\}$, the following inequalities are valid:

$$|\Xi_0(\lambda)| > d > \frac{C}{|\lambda|} > |\Xi(\lambda) - \Xi_0(\lambda)|.$$

Since $r > 0$ can be chosen arbitrary small, we apply Rouché's theorem and obtain the assertion of Lemma 2.4. □

The set $\{\lambda_k\}_{-\infty}^{\infty}$ coincides with the spectrum of the boundary-value problem generated by equations (1.1) for $j = 1, 2, 3$ and the boundary conditions (1.3) and the following condition at $x = a$:

$$\left.\begin{array}{l} y_j(a) = y_{j'}(a) \quad \text{for } j, j' = 1, 2, 3, \\ \displaystyle\sum_{j=1}^{3} y_j'(a) + i\alpha\lambda y_1(a) = 0. \end{array}\right\} \tag{2.25}$$

This problem has the following physical sense: It describes small transverse vibrations of a three-star graph of three inhomogeneous smooth strings damped at the interior vertex(see (30)). This problem also has the following operator interpretation. Denote by A_1 the operator acting in the Hilbert space $H_1 = L_2(0,a) \oplus L_2(0,a) \oplus L_2(0,a) \oplus \mathbb{C}$ with standard inner product $(.,.)_{H_1}$ according to the formulas

$$A_1 \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_1(a) \end{pmatrix} = \begin{pmatrix} -y_1''(x) + q_1(x)y_1(x) \\ -y_2''(x) + q_2(x)y_2(x) \\ -y_3''(x) + q_3(x)y_3(x) \\ \sum_{j=1}^{3} y_j'(a) \end{pmatrix}, \tag{2.26}$$

$$D(A_1) = \left\{ \begin{pmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_1(a) \end{pmatrix} \middle| \begin{array}{l} y_j(x) \in W_2^2(0,a) \quad \text{for } j = 1,2,3, \\ y_j(a) = y_{j'}(a) \quad \text{for } j,j' = 1,2,3, \\ y_1'(0) = y_2'(0) = y_3(0) = 0 \end{array} \right\}. \tag{2.27}$$

Let K and P denote the projectors:

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha I \end{pmatrix}, \quad P = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Clearly $K \geq 0$ and $P \geq 0$. We consider the nonmonic quadratic operator pencil of the form

$$L(\lambda) = \lambda^2 P - i\lambda K - A_1$$

with the domain $D(L(\lambda)) = D(A_1)$ independent of λ and dense in H_1.

Definition 2.2. ((19, Sec.11)) Let $L(\lambda)$ be an operator pencil defined on a complex Hilbert space $\mathcal{H}$. The set of values $\lambda \in \mathbb{C}$ such that $L(\lambda)^{-1}$ exists as a closed bounded linear operator on $\mathcal{H}$ is called the resolvent set $\rho(L)$ of the operator pencil $L(\lambda)$. We denote by $\sigma(L)$ the spectrum of $L(\lambda)$, i.e., the set $\sigma(L) = \mathbb{C} \setminus \rho(L)$. The number $\lambda_0 \in \mathbb{C}$ is said to be an eigenvalue of $L(\lambda)$ if there exists a nonzero vector y_0 (called an eigenvector) such that $L(\lambda)y_0 = 0$. The vectors $y_1, y_2, \ldots, y_{r-1}$ are called corresponding associated vectors if

$$\sum_{s=0}^{n} \frac{1}{s!} \frac{d^s}{d\lambda^s} L(\lambda)\bigg|_{\lambda=\lambda_0} y_{r-s}, \quad n = 1, \ldots, r-1.$$

The number r is called the length of the chain composed of the eigenvector and its associated vectors. The algebraic multiplicity of an eigenvalue is defined as the maximal value of the sum of the lengths of chains corresponding to linearly independent eigenvectors. An eigenvalue is said to be isolated if it has a punctured neighborhood contained in the resolvent set. An isolated eigenvalue λ_0 of finite algebraic multiplicity is said to be normal if the image Im $L(\lambda_0)$ is closed.

It is clear that the set of eigenvalues of the operator pencil $L(\lambda)$ coincides with the spectrum of the boundary-value problem (1.1), (1.3), (2.25).

Theorem 2.5.
1. The spectrum of $L(\lambda)$ consists of normal eigenvalues.
Im $\lambda_k \geq 0$ *for all* $k \in \mathbb{Z}$
3. The eigenvalues of the operator pencil $L(\lambda)$ are nonzero.
4. The spectrum of the boundary-value problem (1.1), (1.3), (2.25) is symmetric with respect to the imaginary axis and symmetrically located eigenvalues possess the same multiplicities.

Proof. Similar to the operator A, one can show that A_1 is self-adjoint in the Hilbert space H_1. We prove that it is strictly positive($A_1 \gg 0$). Let $Y_1 \in D(A_1)$ and $Y \in D(A)$, then integration by parts yields

$$(A_1Y_1, Y_1)_{H_1} = (AY, Y)_H = \sum_{j=1}^{3} \int_0^a (|y_j'(x)|^2 + q_j(x)|y_j(x)|^2)dx.$$

Therefore A_1 is positive. But, if 0 were an eigenvalue of A_1, it should be an eigenvalue of A also. Thus, A_1 is strictly positive. Now taking this together with $K \geq 0$, $P \geq 0$ into account, assertions 1 and 2 follows from (28). Assertion 3 follows from $L_0 = -A_1$. finally, Assertion 4 follows from the identity $\Xi(-\overline{\lambda}) = \overline{\Xi(\lambda)}$. □

We enumerate the zeros of $\Xi(\lambda)$ in the proper way such that (2.24) be valid: 1)$\lambda_{-k} = -\overline{\lambda_k}$ for all not pure imaginary λ_k, 2)Re $\lambda_{k+1} \geq$ Re λ_k, 3) the multiplicities are taken into account, 4) $\lambda_k = \lambda_k^{(0)} + o(1)$. By Lemma 2.4 and assertion 2 of Theorem 2.5, this enumeration is possible. We note that the number of pure imaginary zeros of $\Xi(\lambda)$ is odd due to asymptotics of λ_k and hence this enumeration include the index zero.

Let us recall some definitions from the theory of entire functions.

Definition 2.3. ((15; 17)) An entire function $\omega(\lambda)$ of exponential type $\sigma > 0$ is said to be a function of sine-type if it satisfies the following conditions:

i. all the zeros of $\omega(\lambda)$ lie in a strip $|\text{Im}\,\lambda| < h < \infty$;

ii. for some h_1 and all $\lambda \in \{\lambda : \text{Im}\,\lambda = h_1\}$, the following equalities hold:

$$0 < m \leq |\omega(\lambda)| \leq M < \infty;$$

iii. the type of $\omega(\lambda)$ in the lower half-plane coincides with that in the upper half-plane.

Definition 2.4. ((14, p. 307)) An entire function $\omega(\lambda)$ is said to be of Hermite-Biehler(HB) class if it has no zeros in the closed lower half-plane Im $\lambda \leq 0$, and if

$$\left|\frac{\omega(\lambda)}{\overline{\omega}(\lambda)}\right| < 1 \quad for \ \text{Im}\,\lambda > 0.$$

Here and in the next definition $\overline{\omega}(\lambda)$ denotes the entire function obtained from $\omega(\lambda)$ by replacing the coefficients in its Taylor series by their complex-conjugates, i.e., $\overline{\omega}(\lambda) = \overline{\omega(\overline{\lambda})}$.

Definition 2.5. ((14, p. 313)) An entire function $\omega(\lambda)$ that has no zeros in the open lower half-plane Im $\lambda < 0$ and satisfies the condition

$$\left|\frac{\omega(\lambda)}{\overline{\omega}(\lambda)}\right| \leq 1 \quad for \ \text{Im}\,\lambda > 0$$

is said to be a function of generalized Hermite-Biehler($\overline{HB}$) class.

Lemma 2.6.
1. The function $\Xi(\lambda)$ is of sine-type.
2. The following formula is valid:

$$\Xi(\lambda) = C \prod_{-\infty}^{\infty} \left(1 - \frac{\lambda}{\lambda_k}\right), \tag{2.28}$$

where C is a constant.

Proof. It follows from Lemma 2.4 and from (2.22), (2.23) that $\Xi(\lambda)$ satisfies condition (i) of Definition 2.3. From Lemma 2.3, we conclude that this function satisfies also condition (ii) of Definition 2.3. Using (2.14), it is easy to check up that the types of $\Xi(\lambda)$ in the lower and in the upper half-planes are equal to $3a$ both. Assertion 1 of Lemma 2.6 is proved. Now since by Theorem 2.5 $\lambda_k \neq 0$ for all $k \in \mathbb{Z}$ assertion 2 follows (17). □

Lemma 2.7. *The function $\Xi(\lambda)$ is of generalized Hermite-Biehler class.*

Proof. Let us rearrange the sequence $\{\lambda_k\}_{-\infty}^{\infty}$ into two subsequences $\{\lambda_{m_k}\}_{k=-n}^{n}$ and $\{\lambda_{p_k}\}_{k=-\infty}^{\infty}$ ($m_{-k} = -m_k$ and $p_{-k} = -p_k$ and consequently, $\lambda_{m_{-k}} = -\overline{\lambda}_{m_k}$ and $\lambda_{p_{-k}} = -\overline{\lambda}_{p_k}$) such that $\{\lambda_{m_k}\}_{k=-n,k\neq 0}^{n} \bigcup \{\lambda_{p_k}\}_{k=-\infty}^{\infty} = \{\lambda_k\}_{-\infty}^{\infty}$, $n \leq \infty$, $\operatorname{Im} \lambda_{m_k} = 0$ for all $k \in \mathbb{Z}$, $-n \leq k \leq n$ and $\operatorname{Im} \lambda_{p_k} > 0$ for all $k \in \mathbb{Z}$. Now we can rewrite (2.28) as follows:

$$\Xi(\lambda) = C\widehat{\Xi}(\lambda)\widetilde{\Xi}(\lambda), \tag{2.29}$$

where

$$\widehat{\Xi}(\lambda) = \prod_{k=-n}^{n} \left(1 - \frac{\lambda}{\lambda_{m_k}}\right), \quad \widetilde{\Xi}(\lambda) := \prod_{k=-\infty}^{\infty} \left(1 - \frac{\lambda}{\lambda_{p_k}}\right). \tag{2.30}$$

By virtue of (2.22)-(2.23), we have

$$\left|\operatorname{Im} \frac{1}{\lambda_{p_k}^0}\right| = O\left(\frac{1}{p_k^2}\right).$$

From this and (2.24) it follows that

$$\left|\operatorname{Im} \frac{1}{\lambda_{p_k}}\right| = O\left(\frac{1}{p_k^2}\right).$$

consequently,

$$\sum_{k=-\infty}^{\infty} \left|\operatorname{Im} \frac{1}{\lambda_{p_k}}\right| < \infty.$$

Now M. G. Krein's theorem (14, Chap. 7.2, Theorem 6) implies that

$$\widetilde{\Xi}(\lambda) \in HB. \tag{2.31}$$

It is clear that

$$\left|1 - \frac{\lambda}{\lambda_{p_k}}\right| \leq \left|1 - \frac{\lambda}{\overline{\lambda}_{p_k}}\right|, \quad k \in \mathbb{Z}, \quad \operatorname{Im} \lambda > 0.$$

Together with (2.31) this implies that

$$\left|\frac{\Xi(\lambda)}{\overline{\Xi}(\lambda)}\right| = \prod_{k=-\infty}^{\infty} \left|1 - \frac{\lambda}{\lambda_{p_k}}\right| \left|1 - \frac{\lambda}{\overline{\lambda}_{p_k}}\right|^{-1} \leq 1.$$

Thus, $\Xi(\lambda) \in \overline{HB}$. □

Corollary 2.8. *The sequences $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ and $\{\theta_k\}_{-\infty,k\neq 0}^{\infty} \bigcup \{0\}$ interlace in the following usual sense:*

$$\cdots \leq \theta_{-2} \leq \tau_{-2} \leq \theta_{-1} \leq \tau_{-1} \leq 0 \leq \tau_1 \leq \theta_1 \leq \tau_2 \leq \theta_2 \leq \cdots. \tag{2.32}$$

Proof. This corollary follows from (14, Chap. 7.2, Theorem $3'$) applied to (2.13). □

Theorem 2.9. *The sequences $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ and $\{\theta_k\}_{-\infty}^{\infty}$(we set $\theta_0 = 0$) interlace in the following sense:*

1. $\theta_0 < \tau_1 < \theta_1$.

2. For each simple $\tau_k (k > 1)$, either

$$\theta_{k-1} < \tau_k < \theta_k$$

or

$$\tau_{k-1} < \theta_{k-1} = \tau_k = \theta_k < \tau_{k+1}.$$

3. For each double $\tau_k = \tau_{k+1} (k > 1)$

$$\tau_{k-1} < \theta_{k-1} = \tau_k = \theta_k = \tau_{k+1} = \theta_{k+1} < \tau_{k+2}.$$

4. The multiplicity of each τ_k is ≤ 2.

Proof. By virtue of (2.13), (2.29), (2.30) and the identity $\Xi(-\overline{\lambda}) = \overline{\Xi(\lambda)}$, the entire function $\widetilde{\Xi}(\lambda)$ can be represented as follows:

$$\widetilde{\Xi}(\lambda) = P(\lambda) + i\lambda Q(\lambda),$$

where

$$P(\lambda) = \frac{\varphi_1(\lambda)}{\widehat{\Xi}(\lambda)} = \frac{\widetilde{\Xi}(\lambda) + \widetilde{\Xi}(-\lambda)}{2}, \quad Q(\lambda) = \frac{\varphi_2(\lambda)}{\widehat{\Xi}(\lambda)} = \frac{\widetilde{\Xi}(\lambda) - \widetilde{\Xi}(-\lambda)}{2i\lambda}.$$

Since $\widetilde{\Xi}(\lambda) \in HB$, by N. Meiman's theorem (14, Chap. 7.2, Theorem 3) the zeros $\{\tau_{m_k}\}_{k=-\infty,k\neq 0}^{\infty}$ of $P(\lambda)$ and the zeros $\{0\}\bigcup\{\theta_{p_k}\}_{k=-\infty,p_k\neq 0}^{\infty}$ of $\lambda Q(\lambda)$ ($m_{-k} = -m_k$ and $p_{-k} = -p_k$) interlace in the following strict sense:

$$\cdots < \theta_{p_{-2}} < \tau_{m_{-2}} < \theta_{p_{-1}} < \tau_{m_{-1}} < 0 < \tau_{m_1} < \theta_{p_1} < \tau_{m_2} < \theta_{p_2} < \cdots . \tag{2.33}$$

Now the assertions of Theorem 2.9 easily follow from (2.33), Lemma 2.2 and Corollary 2.8. □

General results on interlacing of the eigenvalues in the case of a star graph with arbitrary number of edges can be found in (29).

Theorem 2.10. *The set $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ of zeros of $\varphi_1(\lambda)$ can be represented as the union of three subsequences $\bigcup_{j=1}^{3}\{\tau_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ which being enumerated in the following way: $\tau_{-k}^{(1)} = -\tau_k^{(1)}$, $\tau_{-k}^{(2)} = -\tau_k^{(3)}$ and $\tau_k^{(j)} \leq \tau_{k+1}^{(j)}$ for $j = 1, 2, 3$, behave asymptotically as follows:*

$$\tau_k^{(1)} = \frac{\pi\left(k - \frac{1}{2}\right)}{a} + \frac{B_1 + B_2}{2\pi\left(k - \frac{1}{2}\right)} + \frac{\beta_k^{(1)}}{k}, \tag{2.34}$$

$$\tau_k^{(j)} = \frac{k\pi + (-1)^j \sin^{-1}\sqrt{\frac{1}{3}}}{a} + \frac{B_1 + B_2 + 2B_3}{4k\pi} + \frac{\beta_k^{(j)}}{k}, \quad j = 2, 3, \tag{2.35}$$

where $\{\beta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$ for $j = 1, 2, 3$.

Proof. In the same way as Lemma 2.4, we can show that the set of zeros $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ can be arranged into three subsequences $\{\tau_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$, $\{\tau_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\tau_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ enumerated in the following way: $\tau_{-k}^{(1)} = -\tau_k^{(1)}$, $\tau_{-k}^{(2)} = -\tau_k^{(3)}$ and $\tau_k^{(j)} \leq \tau_{k+1}^{(j)}$ for $j = 1, 2, 3$ and such that $\{\tau_k\}_{-\infty,k\neq 0}^{\infty} = \bigcup_{j=1}^{3}\{\tau_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, and

$$\tau_k^{(1)} = \frac{\pi\left(k - \frac{1}{2}\right)}{a} + \varepsilon_k^{(1)}, \tag{2.36}$$

$$\tau_k^{(j)} = \frac{k\pi + (-1)^j \sin^{-1}\sqrt{\frac{1}{3}}}{a} + \varepsilon_k^{(j)}, \quad j = 2, 3, \tag{2.37}$$

where $\varepsilon_k^{(j)} = o(1)$ for $j = 1, 2, 3$. It is not difficult to see that

$$\varepsilon_k^{(j)} = O\left(\frac{1}{k}\right), \quad j = 1, 2, 3. \tag{2.38}$$

In fact, we can calculate $\lim_{k\to\infty} k\varepsilon_k^{(j)}a$. Substituting (2.36) into $\varphi_1(\tau_k^{(1)}) = 0$, then from (2.19) and using Paley-Wiener theorem, we have

$$\begin{aligned}\varphi_1(\tau_k^{(1)}) &= (-1)^k\left(3\sin^3\varepsilon_k^{(1)}a - 2\sin\varepsilon_k^{(1)}a\right) \\ &\quad -(-1)^k a(2B_1 + 2B_2 + 3B_3)\frac{\sin^2\varepsilon_k^{(1)}a\cos\varepsilon_k^{(1)}a}{\pi\left(k - \frac{1}{2}\right)} \\ &\quad +(-1)^k a(B_1 + B_2)\frac{\cos^3\varepsilon_k^{(1)}a}{\pi\left(k - \frac{1}{2}\right)} + O\left(\frac{1}{k}\right) = 0.\end{aligned}$$

Since $\lim_{k\to\infty}\cos\varepsilon_k^{(1)}a = 1$, we get $\sin\varepsilon_k^{(1)}a = O\left(\frac{1}{k}\right)$. Thus, $\varepsilon_k^{(1)} = O\left(\frac{1}{k}\right)$. Similarly, we can show that $\varepsilon_k^{(j)} = O\left(\frac{1}{k}\right)$ for $j = 2,3$. Substituting (2.36) into the equation $\varphi_1(\tau_k^{(1)}) = 0$ where $\varphi_1(\lambda)$ is given by (2.19), by expanding the left-hand side of resulting equation in power series and taking into account (2.38) and $\{\omega_1(\tau_k^{(1)})\}_{-\infty,k\neq 0}^{\infty} \in l_2$(see (18, Lemma 1.4.3)), we obtain

$$2\varepsilon_k^{(1)}a - \frac{a(B_1+B_2)}{\pi\left(k-\frac{1}{2}\right)} + \frac{\kappa_k}{k} = 0,$$

where $\{\kappa_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$. Solving this equation we get (2.34). In the same way, we get (2.35). □

General results on interlacing of the eigenvalues in the case of a star graph with arbitrary number of edges can be found in (29) To compare necessary conditions on a sequence to be the spectrum of the boundary-value problem L_0 with the sufficient condition which will be obtained in Section 3, we need more refined asymptotics.

Theorem 2.11. *Let $q_j(x) \in W_2^1(0,a)$. Then the subsequences of Theorem 2.10 behave asymptotically as follows:*

$$\tau_k^{(1)} = \frac{\pi\left(k-\frac{1}{2}\right)}{a} + \frac{B_1+B_2}{2\pi\left(k-\frac{1}{2}\right)} + \frac{\beta_k^{(1)}}{k^2}, \tag{2.39}$$

$$\tau_k^{(j)} = \frac{k\pi + (-1)^j\sin^{-1}\sqrt{\frac{1}{3}}}{a} + \frac{B_1+B_2+2B_3}{4k\pi} + \frac{\beta_k^{(j)}}{k^2}, \quad j = 2,3, \tag{2.40}$$

where $\{\beta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$ for $j = 1,2,3$.

Proof. If $q_j(x) \in W_2^1(0,a)$, twice integrating by parts the formulas of (18, p. 9), we obtain

$$c_j(a,\lambda) = \cos\lambda a + B_j\frac{\sin\lambda a}{\lambda} + D_j\frac{\cos\lambda a}{\lambda^2} + \frac{\upsilon_j(\lambda)}{\lambda^2}, \quad j = 1,2, \tag{2.41}$$

$$c_j'(a,\lambda) = -\lambda\sin\lambda a + B_j\cos\lambda a + D_j'\frac{\sin\lambda a}{\lambda} + \frac{\varrho_j(\lambda)}{\lambda}, \quad j = 1,2, \tag{2.42}$$

$$s_3(a,\lambda) = \frac{\sin\lambda a}{\lambda} - B_3\frac{\cos\lambda a}{\lambda^2} + E\frac{\sin\lambda}{\lambda^3} + \frac{\upsilon_3(\lambda)}{\lambda^3}, \tag{2.43}$$

$$s_3'(a,\lambda) = \cos\lambda a + B_3\frac{\sin\lambda a}{\lambda} + E'\frac{\cos\lambda a}{\lambda^2} + \frac{\varrho_3(\lambda)}{\lambda^2}, \tag{2.44}$$

where D_j, D_j', $j = 1,2$, E and E' are constants and $\upsilon_j(\lambda)$, $\varrho_j(\lambda)$, $j = 1,2,3$ are entire functions of class L^a. Substituting (2.41)-(2.44) into (2.3) we obtain

$$\begin{aligned}\varphi_1(\lambda) &= 3\cos^3\lambda a - 2\cos\lambda a + (2B_1+2B_2+3B_3)\frac{\cos^2\lambda a\sin\lambda a}{\lambda} \\ &\quad -(B_1+B_2)\frac{\sin^3\lambda a}{\lambda} + F_1\frac{\cos^3\lambda a}{\lambda^2} + F_2\frac{\sin^2\lambda a\cos\lambda a}{\lambda^2} + \frac{\omega_3(\lambda)}{\lambda^2},\end{aligned} \tag{2.45}$$

where F_1, F_2 are constants and $\omega_3(\lambda) \in L^{3a}$. Substituting (2.34) into the equation $\varphi_1(\tau_k^{(1)}) = 0$ where $\varphi_1(\lambda)$ is given by (2.45) and by expanding the left-hand side of resulting equation in power series, we get (2.39). Analogously, we obtain (2.40). Theorem 2.11 is proved. □

Remark 2.1. Under the conditions of Theorem 2.11, the spectra $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ of the boundary-value problems L_j for $j = 1,2,3$ behave asymptotically as follows(see (18, p. 75)):

$$\nu_k^{(j)} = \frac{\pi\left(k-\frac{1}{2}\right)}{a} + \frac{B_j}{\pi\left(k-\frac{1}{2}\right)} + \frac{\delta_k^{(j)}}{k^2}, \quad j = 1,2, \tag{2.46}$$

$$\nu_k^{(3)} = \frac{\pi k}{a} + \frac{B_3}{\pi k} + \frac{\delta_k^{(3)}}{k^2}, \tag{2.47}$$

where $\{\delta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}\in l_2$ for $j=1,2,3.$

3 Inverse problem

In the present section, we study the problem of reconstruction of the potential $q(x)=[q_j(x)]_{j=1,2,3}$ from the given spectral characteristics. Let us denote by Q the set of functions $q(x)=[q_j(x)]_{j=1,2,3}$ which satisfy the following conditions:

i. $q_j(x)$, $j=1,2,3$ are real-valued functions from $L_2(0,a)$;

ii. the operator A constructed via (1.6), (1.7) is strictly positive.

Theorem 3.1. *Let the following conditions be valid :*
1. Three sequences $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, $j=1,2,3$ of real numbers are such that

i. $\nu_{-k}^{(j)}=-\nu_k^{(j)}$, $\nu_k^{(j)}<\nu_{k+1}^{(j)}$, $\nu_k^{(j)}\neq 0$ for all $k\in\mathbb{Z}\setminus\{0\}$ and $j=1,2,3$;

ii. $\{\nu_k^{(j)}\}_{-\infty k\neq 0}^{\infty}\bigcap\{\nu_k^{(j')}\}_{-\infty k\neq 0}^{\infty}=\emptyset$ for $j\neq j'$, $j,j'=1,2,3$;

iii.

$$\nu_k^{(j)} = \frac{\pi\left(k-\frac{1}{2}\right)}{a}+\frac{B_j}{\pi\left(k-\frac{1}{2}\right)}+\frac{\delta_k^{(j)}}{k^2},\quad j=1,2, \tag{3.1}$$

$$\nu_k^{(3)} = \frac{\pi k}{a}+\frac{B_3}{\pi k}+\frac{\delta_k^{(3)}}{k^2}, \tag{3.2}$$

where B_j are real constants, $B_j\neq B_{j'}$ for $j\neq j'$ and $\{\delta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}\in l_2$ for $j=1,2,3.$
2. A sequence $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ of real numbers($\tau_{-k}=-\tau_k$, $\tau_k\leq\tau_{k+1}$, $\tau_k\neq 0$ for all $k\in\mathbb{Z}\setminus\{0\}$) can be represented as the union of three subsequences $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}=\bigcup_{j=1}^{3}\{\tau_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ ($\tau_{-k}^{(1)}=-\tau_k^{(1)}$, $\tau_{-k}^{(2)}=-\tau_k^{(3)}$ and $\tau_k^{(j)}\leq\tau_{k+1}^{(j)}$ for $j=1,2,3$) which behave asymptotically as follows:

$$\tau_k^{(1)} = \frac{\pi\left(k-\frac{1}{2}\right)}{a}+\frac{B_1+B_2}{2\pi\left(k-\frac{1}{2}\right)}+\frac{\beta_k^{(1)}}{k^2}, \tag{3.3}$$

$$\tau_k^{(j)} = \frac{k\pi+(-1)^j\xi}{a}+\frac{B_0}{k\pi}+\frac{\beta_k^{(j)}}{k^2},\quad j=2,3, \tag{3.4}$$

where $\xi=\sin^{-1}\sqrt{1/3}$, $B_0=(B_1+B_2+2B_3)/4$ and $\{\beta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}\in l_2$ for $j=1,2,3.$
3. The sequences $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ and $\{\theta_k\}_{-\infty}^{\infty}:=\bigcup_{j=1}^{3}\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}\bigcup\{0\}$($\theta_{-k}=-\theta_k$, $\theta_k<\theta_{k+1}$) interlace in the following strict sense:

$$\cdots<\theta_{-2}<\tau_{-2}<\theta_{-1}<\tau_{-1}<\theta_0=0<\tau_1<\theta_1<\tau_2<\theta_2<\cdots. \tag{3.5}$$

Then there exists a unique function $q(x)=[q_j(x)]_{j=1,2,3}\in Q$ such that the sequence $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ coincides with the spectrum of the boundary-value problem L_0 and the sequences $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ coincides with the spectra of the boundary-value problems L_j for $j=1,2,3$, respectively.

Proof. Denote by

$$\{\rho_k^{(0)}\}_{-\infty,k\neq 0}^{\infty} := \{\frac{\pi k-\xi}{a}\}_{-\infty,k\neq 0}^{\infty}\bigcup\{\frac{\pi k+\xi}{a}\}_{-\infty,k\neq 0}^{\infty},$$
$$\{\rho_k\}_{-\infty,k\neq 0}^{\infty} := \{\tau_k^{(2)}\}_{-\infty,k\neq 0}^{\infty}\bigcup\{\tau_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}.$$

It is possible to enumerate $\{\rho_k^{(0)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\rho_k\}_{-\infty,k\neq 0}^{\infty}$ in the proper way($\rho_{-k}^{(0)} = -\rho_k^{(0)}$, $\rho_k^{(0)} < \rho_{k+1}^{(0)}$ and $\rho_{-k} = -\rho_k$, $\rho_k \leq \rho_{k+1}$). Let us construct the following entire functions:

$$c_j(\lambda) = \prod_1^\infty \left(\frac{a^2}{\pi^2(k-1/2)^2}(\nu_k^{(j)2} - \lambda^2) \right), \quad j = 1, 2, \tag{3.6}$$

$$s_3(\lambda) = a \prod_1^\infty \left(\frac{a^2}{\pi^2 k^2}(\nu_k^{(3)2} - \lambda^2) \right), \tag{3.7}$$

$$\phi_1(\lambda) = \prod_1^\infty \left(\frac{a^2}{\pi^2(k-1/2)^2}(\tau_k^{(1)2} - \lambda^2) \right), \tag{3.8}$$

$$\phi_2(\lambda) = \prod_1^\infty \left(\frac{1}{\rho_k^{(0)2}}(\rho_k^2 - \lambda^2) \right). \tag{3.9}$$

Using Lemma 2.1 of (24), we obtain

$$s_3(\lambda) = \frac{\sin \lambda a}{\lambda} - B_3 \frac{\cos \lambda a}{\lambda^2} + D \frac{\sin \lambda a}{\lambda^3} + \frac{f(\lambda)}{\lambda^3}, \tag{3.10}$$

where D is a constant and $f(\lambda) \in L^a$. In the same way as Lemma 2.1 of (24) we can prove that

$$c_j(\lambda) = \cos \lambda a + B_j \frac{\sin \lambda a}{\lambda} + E^{(j)} \frac{\cos \lambda a}{\lambda^2} + \frac{g_j(\lambda)}{\lambda^2}, \quad j = 1, 2, \tag{3.11}$$

$$\phi_1(\lambda) = \cos \lambda a + \left(\frac{B_1 + B_2}{2} \right) \frac{\sin \lambda a}{\lambda} + F^{(1)} \frac{\cos \lambda a}{\lambda^2} + \frac{h_1(\lambda)}{\lambda^2}, \tag{3.12}$$

$$\phi_2(\lambda) = 3\cos^2 \lambda a - 2 + 3B_0 \frac{\sin 2\lambda a}{\lambda} + F^{(2)} \frac{3\cos^2 \lambda a - 2}{\lambda^2} + \frac{h_2(\lambda)}{\lambda^2}, \tag{3.13}$$

where $E^{(j)}, F^{(j)}$, $j = 1, 2$ are constants and $g_j(\lambda) \in L^a$ for $j = 1, 2$, $h_1(\lambda) \in L^a$ and $h_2(\lambda) \in L^{2a}$. Substituting (3.1) into (3.10)-(3.13), we obtain

$$c_2(\nu_k^{(1)}) = (-1)^k \frac{(B_1 - B_2)a}{\pi\left(k - \frac{1}{2}\right)} + \frac{\zeta_k^{(1)}}{k^2}, \tag{3.14}$$

$$s_3(\nu_k^{(1)}) = \frac{(-1)^{k+1}}{\nu_k^{(1)}} \left(1 + \frac{\zeta_k^{(2)}}{k} \right), \tag{3.15}$$

$$\phi_1(\nu_k^{(1)}) = (-1)^k \frac{(B_1 - B_2)a}{2\pi\left(k - \frac{1}{2}\right)} + \frac{\zeta_k^{(3)}}{k^2}, \tag{3.16}$$

$$\phi_2(\nu_k^{(1)}) = -2 + \frac{\zeta_k^{(4)}}{k}, \tag{3.17}$$

where $\{\zeta_k^{(j)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$ for $j = \overline{1,4}$.

Let us set

$$X_k^{(1)} := \left(\frac{\phi_1(\nu_k^{(1)})\phi_2(\nu_k^{(1)})}{c_2(\nu_k^{(1)})s_3(\nu_k^{(1)})} + \nu_k^{(1)} \sin \nu_k^{(1)} a - B_1 \cos \nu_k^{(1)} a \right). \tag{3.18}$$

It is clear that $X_{-k}^{(1)} = X_k^{(1)}$. Using (3.1), we obtain the asymptotic relation

$$\nu_k^{(1)} \sin \nu_k^{(1)} a - B_1 \cos \nu_k^{(1)} a = (-1)^{k+1} \nu_k^{(1)} \left(1 + \frac{\eta_k}{k} \right), \tag{3.19}$$

where $\{\eta_k\}_{-\infty,k\neq 0}^{\infty} \in l_2$. Taking (3.14)-(3.17) and (3.19) into account, we conclude that $\{X_k^{(1)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$. On the other hand, since $c_1(\lambda)$ is a sine-type function, by virtue of assumption 1(ii), (3.1), (3.2) and (3.5), $\inf_{k\neq p}|\nu_k^{(j)}-\nu_p^{(j)}|>0$ for $j=1,2,3$(and hence the zeros of $\lambda u_j(\lambda)$, $j=1,2$ and $u_3(\lambda)$ are simple), the Lagrange interpolation series

$$c_1(\lambda)\sum_{\substack{-\infty\\ k\neq 0}}^{\infty}\frac{X_k^{(1)}}{\frac{dc_1(\lambda)}{d\lambda}\Big|_{\lambda=\nu_k^{(1)}}(\lambda-\nu_k^{(1)})} \tag{3.20}$$

constructed on the basis of the sequence $\{X_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ defines a function $\varepsilon_1(\lambda)\in L^a$(see (16, Theorem A)). Using this function, we define the even entire function

$$R_1(\lambda)=-\lambda\sin\lambda a+B_1\cos\lambda a+\varepsilon_1(\lambda). \tag{3.21}$$

It follows directly from (3.20) that $\varepsilon_1(\nu_k^{(1)})=X_k^{(1)}$ and hence

$$R_1(\nu_k^{(1)})=\frac{\phi_1(\nu_k^{(1)})\phi_2(\nu_k^{(1)})}{c_2(\nu_k^{(1)})s_3(\nu_k^{(1)})}. \tag{3.22}$$

Let us denote by $\{\mu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ the set of zeros of the function $R_1(\lambda)$. This set is symmetric with respect to the real axis and to the imaginary axis. Hence, we number the zeros in the proper way: $\mu_{-k}=-\mu_k$, $\operatorname{Re}\mu_k\leq\operatorname{Re}\mu_{k+1}$, $k\in\mathbb{N}$ and the multiplicity are taken into account(we shall prove that all μ_k^2 are real and all μ_k are simple except for μ_1, if $\mu_1=\mu_{-1}=0$). It follows from (3.21) that

$$\mu_k^{(1)}=\frac{(k-1)\pi}{a}+\frac{B_1}{k\pi}+\frac{\gamma_k^{(1)}}{k}, \tag{3.23}$$

where $\{\gamma_k^{(1)}\}_{-\infty,k\neq0}^{\infty}\in l_2$.

Lemma 3.2. *The following inequalities are valid:*

$$\mu_1^{(1)2}<\nu_1^{(1)2}<\mu_2^{(1)2}<\nu_2^{(1)2}<\cdots. \tag{3.24}$$

Proof. In the same way as proof of (25, Proposition 2.3), we can show that

$$(-1)^k\frac{\phi_1(\nu_k^{(1)})\phi_2(\nu_k^{(1)})}{c_2(\nu_k^{(1)})s_3(\nu_k^{(1)})}>0. \tag{3.25}$$

Taking into account (3.22), we get

$$(-1)^kR_1(\nu_k^{(1)})>0. \tag{3.26}$$

It follows from (3.26) that between consecutive $\nu_k^{(1)}$'s there is an odd number(with account of multiplicities) of $\mu_k^{(1)}$'s. Suppose that there are three or more of them between $\nu_k^{(1)}$ and $\nu_{k+1}^{(1)}$. Then comparing (3.23) with (3.1), we conclude that there are no $\mu_p^{(1)}$'s between some $\nu_{k'}^{(1)}$ and $\nu_{k'+1}^{(1)}$ where $k\neq k'$, a contradiction. Thus, $\nu_1^{(1)2}<\mu_2^{(1)2}<\nu_2^{(1)2}<\cdots$. If $R_1(0)>0$, then $0<\mu_1^{(1)}<\nu_1^{(1)}$. If $R_1(0)=0$, then $\mu_1^{(1)}=0$. If $R_1(0)<0$, then $\mu_1^{(1)}$ is a pure imaginary number and hence $\mu_1^{(1)2}<\nu_1^{(1)2}$. Lemma 3.2 is proved. ☐

Now the two sequences $\{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ and $\{\mu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ satisfy(due to (3.1), (3.23) and Lemma 3.2) the conditions of (7, Theorem 1.5.4). Thus, it is possible to construct(using the well-known procedure (7, Section 1.5)) a unique real-valued function $q_1(x)\in L_2(0,a)$ such that $\{\nu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ and $\{\mu_k^{(1)}\}_{-\infty,k\neq0}^{\infty}$ are the spectra of the boundary-value problems L_1 and L_1', respectively. In the same way we can construct $q_2(x)$.

Let us construct $q_3(x)$. We set

$$X_k^{(3)} := \nu_k^{(3)} \left(\frac{\phi_1(\nu_k^{(3)})\phi_2(\nu_k^{(3)})}{c_1(\nu_k^{(3)})c_2(\nu_k^{(3)})} - \cos \nu_k^{(3)} a - B_3 \frac{\sin \nu_k^{(3)} a}{\nu_k^{(3)}} \right). \tag{3.27}$$

Clearly $X_{-k}^{(3)} = -X_k^{(3)}$ and in the same way as $\{X_k^{(1)}\}_{-\infty,k\neq 0}^{\infty}$, we can show that $\{X_k^{(3)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$. Note that the function $\lambda s_3(\lambda)$ is a sine-type function and by virtue of assumption 1(ii), (3.1), (3.2) and (3.5), $\inf\limits_{k\neq p} |\nu_k^{(j)} - \nu_p^{(j)}| > 0$ for $j = 1, 2, 3$(and hence the zeros of $\lambda u_j(\lambda)$, $j = 1, 2$ and $u_3(\lambda)$ are simple), and therefore, the Lagrange interpolation series

$$\lambda s_3(\lambda) \sum_{\substack{-\infty \\ k \neq 0}}^{\infty} \frac{X_k^{(3)}}{\nu_k^{(3)} \left.\frac{ds_3(\lambda)}{d\lambda}\right|_{\lambda=\nu_k^{(3)}} (\lambda - \nu_k^{(3)})} \tag{3.28}$$

constructed on the basis of the sequence $\{X_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ defines a function $\varepsilon_3(\lambda) \in L^a$. Let us introduce the even entire function

$$R_3(\lambda) = \cos \lambda a + B_3 \frac{\sin \lambda a}{\lambda} + \frac{\varepsilon_3(\lambda)}{\lambda} \tag{3.29}$$

and denote its zeros by $\{\mu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$. We enumerate this sequence in the proper way. Using (3.29), we get

$$\mu_k^{(3)} = \frac{\pi\left(k - \frac{1}{2}\right)}{a} + \frac{B_3}{\pi\left(k - \frac{1}{2}\right)} + \frac{\gamma_k^{(3)}}{k}. \tag{3.30}$$

where $\{\gamma_k^{(3)}\}_{-\infty,k\neq 0}^{\infty} \in l_2$.

Lemma 3.3. *The following inequalities are valid:*

$$\mu_1^{(3)2} < \nu_1^{(3)2} < \mu_2^{(3)2} < \nu_2^{(3)2} < \cdots. \tag{3.31}$$

Proof. The proof of this Lemma is quite the same as that of Lemma 3.2. □

It follows from (3.31) and the asymptotic relations (3.2) and (3.30) that the sequences $\{\nu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\mu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ satisfy the conditions of (18, Theorem 3.4.1). Thus, it is possible to construct(via the well-known procedure (18, Section 3.4)) a unique real-valued function $q_3(x) \in L_2(0, a)$ such that $\{\nu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ and $\{\mu_k^{(3)}\}_{-\infty,k\neq 0}^{\infty}$ are the spectra of the boundary-value problems L_3 and L_3', respectively.

It is clear that the obtained $q(x) = [q_j(x)]_{j=1,2,3}$ generates the spectra of the boundary-value problems L_j which coincide with $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ for $j = 1, 2, 3$, respectively and the functions $c_1(a, \lambda)$, $c_2(a, \lambda)$ and $s_3(a, \lambda)$ which coincide with $c_1(\lambda)$, $c_2(\lambda)$ and $s_3(\lambda)$ defined by (3.6) for $j = 1, 2$ and (3.7). The set of zeros of values of derivatives $c_1'(a, \lambda)$, $c_2'(a, \lambda)$ and $s_3'(a, \lambda)$ coincide with $\{\mu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$ for $j = 1, 2, 3$ and consequently, $c_1'(a, \lambda)$, $c_2'(a, \lambda)$ and $s_3'(a, \lambda)$ coincide with $R_j(\lambda)$ for $j = 1, 2, 3$($R_2(\lambda)$ is defined analog to $R_1(\lambda)$). Thus, the values of the function $\varphi_1(\lambda)$(defined by (2.3)) at $\lambda = \nu_k^{(j)}$ coincide with

$$\phi_1(\nu_k^{(j)})\phi_2(\nu_k^{(j)})$$

for all $k \in \mathbb{Z}\backslash\{0\}$ and all $j = 1, 2, 3$, i.e., with the corresponding values of the function $\phi_1(\lambda)\phi_2(\lambda)$. This implies that the entire function $\Delta(\lambda) := \varphi_1(\lambda) - \phi_1(\lambda)\phi_2(\lambda)$ of exponential type $3a$ can be represented as follows:

$$\Delta(\lambda) = t(\lambda)c_1(\lambda)c_2(\lambda)s_3(\lambda), \tag{3.32}$$

where $t(\lambda)$ is an entire function. Using (2.20), (3.12) and (3.13) we have

$$\Delta(\lambda) = t(\lambda)\left(\frac{\cos^2\lambda a\sin\lambda a}{\lambda} + (B_1+B_2)\frac{\sin^2\lambda a\cos\lambda a}{\lambda^2} - B_3\frac{\cos^3\lambda a}{\lambda^2} + \frac{\omega_1(\lambda)}{\lambda^2}\right), \tag{3.33}$$

$$\begin{aligned}\phi_1(\lambda)\phi_2(\lambda) &= 3\cos^3\lambda a - 2\cos\lambda a + (2B_1+2B_2+3B_3)\frac{\cos^2\lambda a\sin\lambda a}{\lambda} \\ &\quad -(B_1+B_2)\frac{\sin^3\lambda a}{\lambda} + F_1'\frac{\cos^3\lambda a}{\lambda^2} + F_2'\frac{\sin^2\lambda a\cos\lambda a}{\lambda^2} + \frac{\omega_2(\lambda)}{\lambda^2},\end{aligned} \tag{3.34}$$

where F_1', F_2' are constants, and $\omega_1(\lambda), \omega_2(\lambda) \in L^{3a}$. Comparing (2.45) with (3.33) and (3.34), we obtain

$$\begin{aligned}&t(\lambda)\left(\lambda\cos^2\lambda a\sin\lambda a + (B_1+B_2)\sin^2\lambda a\cos\lambda a - B_3\cos^3\lambda a + \omega_1(\lambda)\right) \\ &\quad = (F_1-F_1')\cos^3\lambda a + (F_2-F_2')\sin^2\lambda a\cos\lambda a + \omega_3(\lambda),\end{aligned} \tag{3.35}$$

where $\omega_3(\lambda) \in L^{3a}$. Since the functions $\cos^3\lambda a$, $\sin^2\lambda a\cos\lambda a$, $\omega_1(\lambda)$ and $\omega_3(\lambda)$ are bounded on the real axis, hence relation (3.35) implies that $t(\lambda) \equiv 0$ and $\varphi_1(\lambda) = \phi_1(\lambda)\phi_2(\lambda)$. Consequently, the sequence $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ coincides with the spectrum of the boundary-value problem L_0. The operator A constructed by (1.6), (1.7) using the obtained $q(x) = [q_j(x)]_{j=1,2,3}$, is strictly positive, because it is self-adjoint and its spectrum is positive. The uniqueness of the solution of the inverse problem follows from the fact that formulas (3.22) and (3.28) establishes one-to-one correspondence between l_2 and L^a(see (16, Theorem A)). Theorem 3.1 is proved. □

Remark 3.1. If condition 1(ii) of theorem 3.1 fails, i.e., the sets $\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}$, $j = 1, 2, 3$ are not pairwise disjoint(consequently, the condition 3 fails too), either the uniqueness or the existence result of mentioned theorem can also fails, for the same reasons as in the case of three spectra(see (24; 10)). If the sequences $\{\tau_k\}_{-\infty,k\neq 0}^{\infty}$ and $\{\theta_k\}_{-\infty}^{\infty} := \bigcup_{j=1}^{3}\{\nu_k^{(j)}\}_{-\infty,k\neq 0}^{\infty}\bigcup\{0\}$ satisfy the statements of Corollary 2.8 and of Theorem 2.9, then the solution of the inverse problem exists but is not unique.

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Competing interests

The authors declare that they have no competing interests.

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