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Distributions of Sum, Difference, Product and Quotient of Independent Non-central Beta Type 3 Variables

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Abstract

Let X and Y be independent random variables, X having a gamma distribution with shape parameter a and Y having a non-central gamma distribution with shape and non-centrality parameters b and δ, respectively. Define $Z = X/(X + 2Y)$. Then, the random variable Z has a non-central beta type 3 distribution, $Z \sim \mathrm{NCB3}(a, b; \delta)$. In this article we derive density functions of sum, difference, product and quotient of two independent random variables each having non-central beta type 3 distribution. These density functions are expressed in series involving first hypergeometric function of Appell.

Keywords: Beta distribution; First hypergeometric function of Appell; Gauss hypergeometric function; Non-central distribution; transformation.

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1 Introduction

The beta type 1 distribution with parameters (a, b) is defined by the probability density function (p.d.f.)

$$\mathrm{B1}(u; a, b) = \frac{u^{a-1}(1-u)^{b-1}}{B(a, b)}, \quad 0 < u < 1, \tag{1.1}$$

where $a > 0$, $b > 0$, and $B(a, b)$ is the beta function defined by

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \mathrm{Re}(a) > 0, \quad \mathrm{Re}(b) > 0.$$

The beta type 1 distribution is well known in Bayesian methodology as a prior distribution on the success probability of a binomial distribution. The random variable V with the p.d.f.

$$\mathrm{B2}(v; a, b) = \frac{v^{a-1}(1+v)^{-(a+b)}}{B(a, b)}, \quad v > 0, \tag{1.2}$$

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where $a > 0$ and $b > 0$, is said to have a beta type 2 distribution with parameters (a, b). Since (1.2) can be obtained from (1.1) by the transformation $V = U/(1 - U)$ some authors call the distribution of V an *inverted beta distribution*. The beta type 1 and beta type 2 are very flexible distributions for positive random variables and have wide applications in statistical analysis, e.g., see [1]. For an in-depth view the reader is referred to an edited volume by Gupta and Nadarajah [2] which contains a collection of essays by various authors covering many different aspects and recent work by [3, 4], and [5]. Systematic treatment of matrix variate generalizations of beta type 1 and beta type 2 distributions is given in [6]. By using the transformation $W = U/(2 - U)$, the beta type 3 p.d.f. is obtained as ([7], [8], [9]),

$$\mathrm{B3}(w; a, b) = \frac{2^a w^{a-1}(1-w)^{b-1}}{B(a,b)(1+w)^{a+b}}, \quad 0 < w < 1, \tag{1.3}$$

where $a > 0$ and $b > 0$.

It is well known that if X and Y are independent random variables having a standard gamma distribution with shape parameters a and b, respectively, then $X/(X+Y) \sim$ B1(a, b), $X/Y \sim$ B2(a, b) and $X/(X + 2Y) \sim$ B3(a, b).

Now, let U be the random variable with a non-central beta type 1 distribution with the p.d.f.

$$\mathrm{NCB1}(u; a, b; \delta) = \frac{\exp(-\delta)\, u^{a-1}(1-u)^{b-1}}{B(a,b)} {}_1F_1\left(a+b; b; \delta(1-u)\right), \quad 0 < u < 1, \tag{1.4}$$

where $a > 0$, $b > 0$, $\delta \geq 0$ and the confluent hypergeometric function ${}_1F_1$ has the integral representation ([10], Eq. 4.2(1)),

$${}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \exp(zt)\, \mathrm{d}t, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0. \tag{1.5}$$

Expanding $\exp(zt)$ in (1.5) and integrating t, the series expansion for ${}_1F_1$ is obtained as

$${}_1F_1(a; c; z) = \sum_{j=0}^{\infty} \frac{\Gamma(c)\Gamma(a+j)}{\Gamma(a)\Gamma(c+j)} \frac{z^j}{j!}. \tag{1.6}$$

The non-central beta type 1 distribution is used in computing power of several test statistics. Recently, Miranda De Sá in [11] has shown that the sampling distribution of coherence estimate between one random and one periodic signal is type 1 non-central beta (also see [12]). This distribution also appears in statistical discrimination and sequential testing of nested linear hypothesis. Nadarajah in [13] has derived distributions of sum, product, and ratio of two independent non-central beta type 1 variables. By making the transformation $V = U/(1 - U)$ in (1.4), the non-central beta type 2 p.d.f. is derived as

$$\mathrm{NCB2}(v; a, b; \delta) = \frac{\exp(-\delta)\, v^{a-1}(1+v)^{-(a+b)}}{B(a,b)} {}_1F_1\left(a+b; b; \frac{\delta}{1+v}\right), \quad v > 0, \tag{1.7}$$

where $a > 0$, $b > 0$ and $\delta \geq 0$. Further, transforming $W = U/(2 - U)$ in (1.4), the non-central beta type 3 p.d.f. is obtained as

$$\mathrm{NCB3}(w; a, b; \delta) = \frac{2^a \exp(-\delta)\, w^{a-1}(1-w)^{b-1}}{B(a,b)(1+w)^{a+b}} {}_1F_1\left(a+b; b; \frac{\delta(1-w)}{1+w}\right), \quad 0 < w < 1, \tag{1.8}$$

where $a > 0$, $b > 0$ and $\delta \geq 0$.

In this article, we give distributions of sum, difference, product and quotient of two independent random variables both having non-central beta type 3 distribution. Some of these results in the central case are available in [14].

2 Some known definitions and results

The first hypergeometric function of Appell is defined in a series form as

$$F_1(a, b_1, b_2; c; z_1, z_2) = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \frac{(a)_{r_1+r_2}(b_1)_{r_1}(b_2)_{r_1}}{(c)_{r_1+r_2}} \frac{z_1^{r_1} z_2^{r_2}}{r_1!\, r_2!}, \tag{2.1}$$

where $|z_1| < 1$ and $|z_2| < 1$ and the Pochhammer symbol $(q)_n$ represents the rising factorial:

$$(q)_n = \frac{\Gamma(q+n)}{\Gamma(q)} = q\,(q+1)\cdots(q+n-1).$$

From (2.1), it follows that

$$\begin{aligned} F_1(a, b_1, b_2; c; z_1, z_2) &= \sum_{r_1=0}^{\infty} \frac{(a)_{r_1}(b_1)_{r_1}}{(c)_{r_1}} \frac{z_1^{r_1}}{r_1!}\ {}_2F_1(a + r_1, b_2; c + r_1; z_2) \\ &= \sum_{r_2=0}^{\infty} \frac{(a)_{r_2}(b_2)_{r_2}}{(c)_{r_2}} \frac{z_2^{r_2}}{r_2!}\ {}_2F_1(a + r_2, b_1; c + r_2; z_1), \end{aligned} \tag{2.2}$$

where ${}_2F_1$ is the Gauss hypergeometric function. Further, writing

$$\frac{(a)_{r_1+r_2}}{(c)_{r_1+r_2}} = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 v^{a+r_1+r_2-1}(1-v)^{c-a-1}\,\mathrm{d}v, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0$$

and

$$\sum_{r_i=0}^{\infty} \frac{(b_i)_{r_i}(vz_i)^{r_i}}{r_i!} = (1 - vz_i)^{-b_i}, \quad |vz_i| < 1, \quad i = 1, 2$$

in (2.1), one can derive an integral representation of the first hypergeometric function of Appell as

$$\begin{aligned} F_1(a, b_1, b_2; c; z_1, z_2) &= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 \frac{v^{a-1}(1-v)^{c-a-1}\,\mathrm{d}v}{(1-vz_1)^{b_1}(1-vz_2)^{b_2}}, \\ &\qquad |z_1| < 1, \quad |z_2| < 1, \quad \mathrm{Re}(c) > \mathrm{Re}(a) > 0. \end{aligned} \tag{2.3}$$

Further, for $c = b_1 + b_2$, F_1 reduces to a Gauss hypergeometric function. That is

$$F_1(a, b_1, b_2; b_1 + b_2; z_1, z_2) = (1 - z_2)^{-a}\,{}_2F_1\left(a, b_1; b_1 + b_2; \frac{z_1 - z_2}{1 - z_2}\right).$$

From the definition, it is easy to see that

$$\begin{aligned} F_1(a, b_1, b_2; c; z_1, z_2) &= {}_2F_1(a, b_2; c; z_2), \quad \text{if} \quad b_1 = 0 \quad \text{or} \quad z_1 = 0, \\ &= {}_2F_1(a, b_1; c; z_1), \quad \text{if} \quad b_2 = 0 \quad \text{or} \quad z_2 = 0 \end{aligned} \tag{2.4}$$

and

$$F_1(a, b_1, b_2; c; z, z) = {}_2F_1(a, b_1 + b_2; c; z).$$

The following representations of the function $F_1(a, b_1, b_2; c; z_1, z_2)$ facilitate calculations of F_1 for different values of parameters:

$$\begin{aligned} F_1(a, b_1, b_2; c; z_1, z_2) &= (1 - z_1)^{-b_1}(1 - z_2)^{-b_2} F_1\left(c - a, b_1, b_2; c; -\frac{z_1}{1 - z_1}, -\frac{z_2}{1 - z_2}\right) \\ &= (1 - z_1)^{-a} F_1\left(a, c - b_1 - b_2, b_2; c; -\frac{z_1}{1 - z_1}, -\frac{z_2 - z_1}{1 - z_1}\right) \\ &= (1 - z_1)^{c-a-b_1}(1 - z_2)^{-b_2} F_1\left(c - a, c - b_1 - b_2, b_2; c; z_1; -\frac{z_2 - z_1}{1 - z_2}\right). \end{aligned}$$

For further insight into these functions the reader is referred to [15], and [16].

To derive our distributional results we essentially need the following result given in [13].

Theorem 2.1. *Let X_1 and X_2 be independent random variables having support in $(0,1)$ and densities f_1 and f_2, respectively. Then, the densities of $S = X_1+X_2$, $D = X_1-X_2$, $P = X_1X_2$ and $Q = X_2/X_1$ can be expressed as*

$$f_S(s) = \begin{cases} \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t, & \textit{if} \quad 0 < s < 1, \\ \\ \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t, & \textit{if} \quad 1 < s < 2, \end{cases}$$

$$f_D(d) = \begin{cases} \int_0^{1+d} f_1(t) f_2(t-d)\,\mathrm{d}t, & \textit{if} \quad -1 < d < 0, \\ \\ \int_0^{1-d} f_1(d+t) f_2(t)\,\mathrm{d}t, & \textit{if} \quad 0 < d < 1, \end{cases}$$

$$f_P(p) = \int_p^1 \frac{f_1(t)}{t} f_2\left(\frac{p}{t}\right)\mathrm{d}t, \quad \textit{if} \quad 0 < p < 1,$$

$$f_Q(q) = \begin{cases} \int_0^1 t f_1(t) f_2(qt)\,\mathrm{d}t, & \textit{if} \quad 0 < q < 1, \\ \\ \int_0^{1/q} t f_1(t) f_2(qt)\,\mathrm{d}t, & \textit{if} \quad q > 1, \end{cases}$$

respectively.

3 Densities of sum, difference, product and quotient

In this section, we give densities of the sum, difference, product and quotient of two independent random variables both having non-central beta type 3 distribution.

Using (1.8) and the series expansion of ${}_1F_1$, the p.d.f. of W, $W \sim \mathrm{NCB3}(a_i, b_i; \delta_i)$, can be written as

$$f_i(w) = \frac{2^{a_i} w^{a_i-1}(1-w)^{b_i-1}}{\Gamma(a_i)(1+w)^{a_i+b_i}} \exp(-\delta) \sum_{j=0}^{\infty} \frac{\Gamma(a_i+b_i+j)}{\Gamma(b_i+j) j!} \left(\frac{1-w}{1+w}\delta_i\right)^j, \tag{3.1}$$

where $0 < w < 1$, $a_i > 0$, $b_i > 0$ and $\delta_i \geq 0$.

Theorem 3.1. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{NCB3}(a_i, b_i; \delta_i)$, $i = 1, 2$. Further, assume that a_1, a_2, b_1 and b_2 are all positive integers. Then, the p.d.f. of $S = X_1 + X_2$ is derived as*

$$\begin{aligned} f_S(s) = {} & \frac{2^{a_1+a_2} \exp(-\delta_1 - \delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{s^{a_1+a_2-1}}{(1+s)^{a_2+b_2}} \sum_{j,k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j \delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k) j! k!} \frac{1}{(1+s)^k} \\ & \times \sum_{r_1=0}^{b_1+j-1} \sum_{r_2=0}^{b_2+k-1} \binom{b_1+j-1}{r_1} \binom{b_2+k-1}{r_2} (-s)^{r_1+r_2} \frac{\Gamma(a_1+r_1)\Gamma(a_2+r_2)}{\Gamma(a_1+a_2+r_1+r_2)} \\ & \times F_1\left(a_1+r_1, a_1+b_1+j, a_2+b_2+k; a_1+a_2+r_1+r_2; -s, \frac{s}{1+s}\right), \end{aligned}$$

for $0 < s < 1$*, and*

$$f_S(s) = \frac{\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{(2-s)^{b_1+b_2-1}}{2^{b_2-a_1} s^{a_1+b_1}} \sum_{j,k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \frac{(2-s)^{j+k}}{2^k s^j}$$
$$\times \sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{a_2-1} \binom{a_1-1}{r_1}\binom{a_2-1}{r_2} [-(2-s)]^{r_1+r_2} \frac{\Gamma(b_1+r_1+j)\Gamma(b_2+r_2+k)}{\Gamma(b_1+b_2+r_1+r_2+j+k)}$$
$$\times F_1\left(b_2+r_2+k, a_1+b_1+j, a_2+b_2+k; b_1+b_2+r_1+r_2+j+k; -\frac{2-s}{s}, \frac{2-s}{2}\right),$$

for $1 < s < 2$*.*

Proof. From Theorem 2.1, the p.d.f. of S is given by

$$f_S(s) = \begin{cases} \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t, & \text{for} \quad 0 < s < 1, \\ \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t, & \text{for} \quad 1 < s < 2, \end{cases}$$

and therefore in our derivation we consider cases $0 < s < 1$ and $1 < s < 2$ separately. Using (3.1), the p.d.f. of S, for $0 < s < 1$, is derived as

$$\begin{aligned} f_S(s) &= \int_0^s f_1(t) f_2(s-t)\,\mathrm{d}t \\ &= s\int_0^1 f_1(sw) f_2(s(1-w))\,\mathrm{d}w \\ &= \frac{2^{a_1+a_2}\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \frac{s^{a_1+a_2-1}}{(1+s)^{a_2+b_2+k}} \\ &\quad\times \int_0^1 \frac{w^{a_1-1}(1-sw)^{b_1+j-1}(1-w)^{a_2-1}[1-s(1-w)]^{b_2+k-1}}{(1+sw)^{a_1+b_1+j}\,[1-sw/(1+s)]^{a_2+b_2+k}}\,\mathrm{d}w. \end{aligned} \tag{3.2}$$

Now, expanding $(1-sw)^{b_1+j-1}$ and $[1-s(1-w)]^{b_2+k-1}$ in terms of binomial series, namely,

$$(1-sw)^{b_1+j-1} = \sum_{r_1=0}^{b_1+j-1} \binom{b_1+j-1}{r_1} (-sw)^{r_1},$$

and

$$[1-s(1-w)]^{b_2+k-1} = \sum_{r_2=0}^{b_2+k-1} \binom{b_2+k-1}{r_2} [-s(1-w)]^{r_2},$$

the integral in (3.2) is evaluated as

$$\begin{aligned} &\sum_{r_1=0}^{b_1+j-1}\sum_{r_2=0}^{b_2+k-1} \binom{b_1+j-1}{r_1}\binom{b_2+k-1}{r_2} (-s)^{r_1+r_2} \\ &\quad\times \int_0^1 \frac{w^{a_1+r_1-1}(1-w)^{a_2+r_2-1}}{(1+sw)^{a_1+b_1+j}\,[1-sw/(1+s)]^{a_2+b_2+k}}\,\mathrm{d}w \\ &= \sum_{r_1=0}^{b_1+j-1}\sum_{r_2=0}^{b_2+k-1} \binom{b_1+j-1}{r_1}\binom{b_2+k-1}{r_2} (-s)^{r_1+r_2} \frac{\Gamma(a_1+r_1)\Gamma(a_2+r_2)}{\Gamma(a_1+a_2+r_1+r_2)} \\ &\quad\times F_1\left(a_1+r_1, a_1+b_1+j, a_2+b_2+k; a_1+a_2+r_1+r_2; -s, \frac{s}{1+s}\right), \end{aligned} \tag{3.3}$$

where the last line has been obtained by using (2.3). Now, substituting (3.3) in (3.2), we get the desired result. Similarly, the p.d.f. of S, for $1 < s < 2$, is obtained as

$$\begin{aligned}
f_S(s) &= \int_0^{2-s} f_1(s-1+t) f_2(1-t)\,\mathrm{d}t \\
&= (2-s)\int_0^1 f_1(s-1+(2-s)w) f_2(1-(2-s)w)\,\mathrm{d}w \\
&= \frac{2^{a_1+a_2}\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \frac{(2-s)^{b_1+b_2+j+k-1}}{2^{a_2+b_2+k}s^{a_1+b_1+j}} \\
&\quad\times \int_0^1 \frac{w^{b_2+k-1}(1-w)^{b_1+j-1}[1-(2-s)w]^{a_2-1}[1-(2-s)(1-w)]^{a_1-1}}{[1+(2-s)w/s]^{a_1+b_1+j}[1-(2-s)w/2]^{a_2+b_2+k}}\,\mathrm{d}w.
\end{aligned}$$

Replacing $[1-(2-s)(1-w)]^{a_1-1}$ and $[1-(2-s)w]^{a_2-1}$ by their respective binomial expansions, namely,

$$[1-(2-s)(1-w)]^{a_1-1} = \sum_{r_1=0}^{a_1-1} \binom{a_1-1}{r_1} [-(2-s)(1-w)]^{r_1}$$

and

$$[1-(2-s)w]^{a_2-1} = \sum_{r_2=0}^{a_2-1} \binom{a_2-1}{r_2} [-(2-s)w]^{r_2}$$

and integrating the resulting expression using (2.3), we get the desired result. □

Corollary 3.2. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{B3}(a_i, b_i)$, $i = 1, 2$. Further, assume that a_1, a_2, b_1 and b_2 are all positive integers. Then, the p.d.f. of $S = X_1 + X_2$ is derived as*

$$\begin{aligned}
f_S(s) &= \frac{2^{a_1+a_2}\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{\Gamma(a_1)\Gamma(b_1)\Gamma(a_2)\Gamma(b_2)} \frac{s^{a_1+a_2-1}}{(1+s)^{a_2+b_2}} \\
&\quad\times \sum_{r_1=0}^{b_1-1}\sum_{r_2=0}^{b_2-1} \binom{b_1-1}{r_1}\binom{b_2-1}{r_2} (-s)^{r_1+r_2} \frac{\Gamma(a_1+r_1)\Gamma(a_2+r_2)}{\Gamma(a_1+a_2+r_1+r_2)} \\
&\quad\times F_1\left(a_1+r_1, a_1+b_1, a_2+b_2; a_1+a_2+r_1+r_2; -s, \frac{s}{1+s}\right), \quad 0 < s < 1
\end{aligned}$$

and

$$\begin{aligned}
f_S(s) &= \frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{\Gamma(a_1)\Gamma(b_1)\Gamma(a_2)\Gamma(b_2)} \frac{(2-s)^{b_1+b_2-1}}{2^{b_2-a_1}s^{a_1+b_1}} \\
&\quad\times \sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{a_2-1} \binom{a_1-1}{r_1}\binom{a_2-1}{r_2} [-(2-s)]^{r_1+r_2} \frac{\Gamma(b_1+r_1)\Gamma(b_2+r_2)}{\Gamma(b_1+b_2+r_1+r_2)} \\
&\quad\times F_1\left(b_2+r_2, a_1+b_1, a_2+b_2; b_1+b_2+r_1+r_2; -\frac{2-s}{s}, \frac{2-s}{2}\right), \quad 1 < s < 2.
\end{aligned}$$

Theorem 3.3. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{NCB3}(a_i, b_i; \delta_i)$, $i = 1, 2$. Further, assume that a_1, a_2, b_1 and b_2 are all positive integers. Then, the p.d.f. of $D = X_1 - X_2$ is* *derived as*

$$f_D(d) = \frac{2^{a_1+a_2}\exp\left(-\delta_1-\delta_2\right)}{\Gamma(a_1)\Gamma(a_2)}\frac{(1+d)^{a_1+b_2-1}}{(1-d)^{a_2+b_2}}\sum_{j,k=0}^{\infty}\frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!}$$
$$\times\frac{(1+d)^k}{(1-d)^k}\sum_{r_1=0}^{b_1+j-1}\sum_{r_2=0}^{a_2-1}\binom{b_1+j-1}{r_1}\binom{a_2-1}{r_2}[-(1+d)]^{r_1+r_2}\frac{\Gamma(a_1+r_1)\Gamma(b_2+r_2+k)}{\Gamma(a_1+b_2+k+r_1+r_2)}$$
$$\times F_1\left(a_1+r_1, a_1+b_1+j, a_2+b_2+k; a_1+b_2+r_1+r_2+k; -(1+d), -\frac{1+d}{1-d}\right)$$

for $-1<d<0$*, and*

$$f_D(d) = \frac{2^{a_1+a_2}\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{(1-d)^{a_2+b_1-1}}{(1+d)^{a_1+b_1}}\sum_{j,k=0}^{\infty}\frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!}\frac{(1-d)^j}{(1+d)^j}$$
$$\times\sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{b_2+k-1}\binom{a_1-1}{r_1}\binom{b_2+k-1}{r_2}[-(1-d)]^{r_1+r_2}\frac{\Gamma(a_2+r_2)\Gamma(b_1+r_1+j)}{\Gamma(a_2+b_1+j+r_1+r_2)}$$
$$\times F_1\left(a_2+r_2, a_1+b_1+j, a_2+b_2+k; a_2+b_1+r_1+r_2+j; -\frac{1-d}{1+d}, -(1-d)\right)$$

for $0<d<1$.

Proof. Theorem 2.1 states the p.d.f. of D as

$$f_D(d) = \begin{cases}\int_0^{1+d} f_1(t)f_2(t-d)\,\mathrm{d}t, & \text{for } -1<d<0,\\ \\ \int_0^{1-d} f_1(d+t)f_2(t)\,\mathrm{d}t, & \text{for } 0<d<1,\end{cases}$$

and therefore we consider cases $-1<d<0$ and $0<d<1$ separately. Using the non-central beta type 3 p.d.f. given in (3.1), the p.d.f. of D, for $-1<d<0$, is derived as

$$\begin{aligned}f_D(d) &= \int_0^{1+d} f_1(t)f_2(t-d)\,\mathrm{d}t\\
&= (1+d)\int_0^1 f_1((1+d)w)f_2((1+d)w-d)\,\mathrm{d}w\\
&= \frac{2^{a_1+a_2}\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)}\frac{(1+d)^{a_1+b_2-1}}{(1-d)^{a_2+b_2}}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!}\frac{(1+d)^k}{(1-d)^k}\\
&\quad\times\int_0^1\frac{w^{a_1-1}(1-w)^{b_2+k-1}[1-(1+d)w]^{b_1+j-1}[1-(1+d)(1-w)]^{a_2-1}}{[1+(1+d)w]^{a_1+b_1+j}[1+(1+d)w/(1-d)]^{a_2+b_2+k}}\mathrm{d}w.\end{aligned} \tag{3.4}$$

Writing

$$\begin{aligned}&[1-(1+d)w]^{b_1+j-1}[1-(1+d)(1-w)]^{a_2-1}\\
&\quad=\sum_{r_1=0}^{b_1+j-1}\sum_{r_2=0}^{a_2-1}\binom{b_1+j-1}{r_1}\binom{a_2-1}{r_2}[-(1+d)]^{r_1+r_2}w^{r_1}(1-w)^{r_2},\end{aligned}$$

and applying (2.3), the above integral is evaluated as

$$\begin{aligned}&\sum_{r_1=0}^{b_1+j-1}\sum_{r_2=0}^{a_2-1}\binom{b_1+j-1}{r_1}\binom{a_2-1}{r_2}[-(1+d)]^{r_1+r_2}\frac{\Gamma(a_1+r_1)\Gamma(b_2+r_2+k)}{\Gamma(a_1+b_2+k+r_1+r_2)}\\
&\times F_1\left(a_1+r_1, a_1+b_1+j, a_2+b_2+k; a_1+b_2+k+r_1+r_2; -(1+d), -\frac{1+d}{1-d}\right).\end{aligned} \tag{3.5}$$

Finally, substituting (3.5) in (3.4), we get the desired result. Using (3.1) and Theorem 2.1, the p.d.f. of D, for $0 < d < 1$, is derived as

$$\begin{aligned}
f_D(d) &= \int_0^{1-d} f_1(d+t) f_2(t)\,\mathrm{d}t \\
&= (1-d)\int_0^1 f_1(d+(1-d)w) f_2((1-d)w)\,\mathrm{d}w \\
&= \frac{2^{a_1+a_2}\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \frac{(1-d)^{a_2+b_1+j-1}}{(1+d)^{a_1+b_1+j}} \\
&\quad \times \int_0^1 \frac{w^{a_2-1}(1-w)^{b_1+j-1}[1-(1-d)(1-w)]^{a_1-1}[1-(1-d)w]^{b_2+k-1}}{[1+(1-d)w/(1+d)]^{a_1+b_1+j}[1+(1-d)w]^{a_2+b_2+k}}\mathrm{d}w. \qquad (3.6)
\end{aligned}$$

Now, expanding $[1-(1-d)(1-w)]^{a_1-1}$ and $[1-(1-d)w]^{b_2+k-1}$ using binomial theorem, the above integral is expressed as

$$\begin{aligned}
&\sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{b_2+k-1} \binom{a_1-1}{r_1}\binom{b_2+k-1}{r_2} [-(1-d)]^{r_1 r_2} \\
&\times \int_0^1 \frac{w^{a_2+r_2-1}(1-w)^{b_1+r_1+j-1}}{[1+(1-d)w/(1+d)]^{a_1+b_1+j}[1+(1-d)w]^{a_2+b_2+k}}\,\mathrm{d}w \\
&= \sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{b_2+k-1} \binom{a_1-1}{r_1}\binom{b_2+k-1}{r_2} [-(1-d)]^{r_1+r_2} \frac{\Gamma(a_2+r_2)\Gamma(b_1+r_1+j)}{\Gamma(a_2+b_1+j+r_1+r_2)} \\
&\times F_1\left(a_2+r_2, a_1+b_1+j, a_2+b_2+k; a_2+b_1+r_1+r_2+j; -\frac{1-d}{1+d}, -(1-d)\right), \qquad (3.7)
\end{aligned}$$

where the last line has been obtained by using (2.3). Finally, substitution for the integral from (3.7) in (3.6) yields the desired result. □

Corollary 3.4. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{B3}(a_i, b_i)$, $i = 1, 2$. Further, assume that a_1, a_2, b_1 and b_2 are all positive integers. Then, the p.d.f. of $D = X_1 - X_2$ is derived as*

$$\begin{aligned}
f_D(d) &= \frac{2^{a_1+a_2}\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1)\Gamma(b_2)} \frac{(1+d)^{a_1+b_2-1}}{(1-d)^{a_2+b_2}} \\
&\quad \times \sum_{r_1=0}^{b_1-1}\sum_{r_2=0}^{a_2-1} \binom{b_1-1}{r_1}\binom{a_2-1}{r_2} [-(1+d)]^{r_1+r_2} \frac{\Gamma(a_1+r_1)\Gamma(b_2+r_2)}{\Gamma(a_1+b_2+r_1+r_2)} \\
&\quad \times F_1\left(a_1+r_1, a_1+b_1, a_2+b_2; a_1+b_2+r_1+r_2; -(1+d), -\frac{1+d}{1-d}\right), \quad -1 < d < 0
\end{aligned}$$

and

$$\begin{aligned}
f_D(d) &= \frac{2^{a_1+a_2}\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1)\Gamma(b_2)} \frac{(1-d)^{a_2+b_1-1}}{(1+d)^{a_1+b_1}} \\
&\quad \times \sum_{r_1=0}^{a_1-1}\sum_{r_2=0}^{b_2-1} \binom{a_1-1}{r_1}\binom{b_2-1}{r_2} [-(1-d)]^{r_1+r_2} \frac{\Gamma(a_2+r_2)\Gamma(b_1+r_1)}{\Gamma(a_2+b_1+r_1+r_2)} \\
&\quad \times F_1\left(a_2+r_2, a_1+b_1, a_2+b_2; a_2+b_1+r_1+r_2; -\frac{1-d}{1+d}, -(1-d)\right), \quad 0 < d < 1.
\end{aligned}$$

Theorem 3.5. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{NCB3}(a_i, b_i; \delta_i)$, $i = 1, 2$. Further, assume that a_1 is a positive integer. Then, the p.d.f. of $P = X_1 X_2$ can be expressed as*

$$f_P(p) = \frac{\exp(-\delta_1 - \delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{p^{a_2-1}(1-p)^{b_1+b_2-1}}{2^{b_1-a_2}(1+p)^{a_2+b_2}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j \delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k) j! k!}$$
$$\times \frac{(1-p)^{j+k}}{2^j(1+p)^k} \sum_{r=0}^{a_1} \binom{a_1}{r} [-(1-p)]^r \frac{\Gamma(b_1+r+j)\Gamma(b_2+k)}{\Gamma(b_1+b_2+r+j+k)}$$
$$\times F_1\left(b_1+r+j, a_1+b_1+j, a_2+b_2+k; b_1+b_2+r+j+k; \frac{1-p}{2}, -\frac{1-p}{1+p}\right), \quad 0 < p < 1.$$

Proof. Using (3.1) and Theorem 2.1, the p.d.f. of P is derived as

$$\begin{aligned} f_P(p) &= \int_p^1 \frac{f_1(t)}{t} f_2\left(\frac{p}{t}\right) \mathrm{d}t \\ &= (1-p)\int_0^1 \frac{f_1(1-(1-p)v)}{1-(1-p)v} f_2\left(\frac{p}{1-(1-p)v}\right) \mathrm{d}v \\ &= \frac{\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{p^{a_2-1}(1-p)^{b_1+b_2-1}}{2^{b_1-a_2}(1+p)^{a_2+b_2}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \\ &\quad \times \frac{(1-p)^{j+k}}{2^j(1+p)^k} \int_0^1 \frac{v^{b_1+j-1}(1-v)^{b_2+k-1}[1-(1-p)v]^{a_1}}{[1-(1-p)v/2]^{a_1+b_1+j}[1+(1-p)v/(1+p)]^{a_2+b_2+k}} \mathrm{d}v. \end{aligned}$$

Now, expansion of $[1-(1-p)v]^{a_1}$ in binomial series and evaluation of the resulting expression using (2.3) yield the desired result. □

Corollary 3.6. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{B3}(a_i, b_i)$, $i = 1, 2$. Further, assume that a_1 is a positive integer. Then, the p.d.f. of $P = X_1 X_2$ can be expressed as*

$$f_P(p) = \frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{\Gamma(a_1)\Gamma(b_1)\Gamma(a_2)} \frac{p^{a_2-1}(1-p)^{b_1+b_2-1}}{2^{b_1-a_2}(1+p)^{a_2+b_2}} \sum_{r=0}^{a_1} \binom{a_1}{r} [-(1-p)]^r \frac{\Gamma(b_1+r)}{\Gamma(b_1+b_2+r)}$$
$$\times F_1\left(b_1+r, a_1+b_1, a_2+b_2; b_1+b_2+r; \frac{1-p}{2}, -\frac{1-p}{1+p}\right), \quad 0 < p < 1.$$

Theorem 3.7. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{NCB3}(a_i, b_i; \delta_i)$, $i = 1, 2$. Further, assume that b_1 and b_2 are positive integers. Then, the p.d.f to $Q = X_1/X_2$ is given as*

$$f_Q(q) = \frac{\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{2^{a_2-b_1} q^{a_2-1}}{(1+q)^{a_2+b_2}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k) j!\, k!}$$
$$\times \frac{1}{2^j(1+q)^k} \sum_{r=0}^{b_2+k-1} \binom{b_2+k-1}{r} \frac{\Gamma(a_1+a_2+r)\Gamma(b_1+j)}{\Gamma(a_1+b_1+a_2+r+j)} q^r$$
$$\times F_1\left(b_1+j, a_1+b_1+j, a_2+b_2+k; a_1+b_1+a_2+j+r; \frac{1}{2}, \frac{q}{1+q}\right), \quad 0 < q < 1$$

and

$$f_Q(q) = \frac{\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{2^{a_1-b_2}}{(q+1)^{a_1+b_1}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k) j!\, k!}$$
$$\times \frac{1}{2^k(1+q)^j} \sum_{r=0}^{b_1+j-1} \binom{b_1+j-1}{r} \frac{\Gamma(a_1+a_2+r)\Gamma(b_2+k)}{\Gamma(a_1+a_2+b_2+r+k)} q^{-r}$$
$$\times F_1\left(b_2+k, a_1+b_1+j, a_2+b_2+k; a_1+a_2+b_2+r+k; \frac{1}{q+1}, \frac{1}{2}\right), \quad q > 1.$$

Proof. Using (3.1) and Theorem 2.1, the p.d.f. of Q, for $0 < q < 1$, is derived as

$$\begin{aligned}
f_Q(q) &= \int_0^1 t f_1(t) f_2(qt)\,\mathrm{d}t \\
&= \frac{2^{a_1+a_2}\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \\
&\quad \times q^{a_2-1}\int_0^1 \frac{t^{a_1+a_2-1}(1-t)^{b_1+j-1}(1-qt)^{b_2+k-1}}{(1+t)^{a_1+b_1+j}(1+qt)^{a_2+b_2+k}}\,\mathrm{d}t \\
&= \frac{\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{q^{a_2-1}}{2^{b_1-a_2}(1+q)^{a_2+b_2}} \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \\
&\quad \times \frac{1}{2^j(1+q)^k}\int_0^1 \frac{w^{b_1+j-1}(1-w)^{a_1+a_2-1}[1-q(1-w)]^{b_2+k-1}}{(1-w/2)^{a_1+b_1+j}[1-qw/(q+1)]^{a_2+b_2+k}}\,\mathrm{d}w.
\end{aligned}$$

Now, applying binomial theorem to $[1-q(1-w)]^{b_2+k-1}$ and integrating with respect to w using (2.3), the final expression for $f_Q(q)$, for $0 < q < 1$, is obtained. The p.d.f. of Q, for $q > 1$, is derived as

$$\begin{aligned}
f_Q(q) &= \int_0^{1/q} t f_1(t) f_2(qt)\ \mathrm{d}t \\
&= q^{-2}\int_0^1 u f_1\left(\frac{u}{q}\right) f_2(u)\ \mathrm{d}u \\
&= \frac{2^{a_1+a_2}\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \\
&\quad \times q^{-(a_1+1)}\int_0^1 \frac{u^{a_1+a_2-1}(1-u/q)^{b_1+j-1}(1-u)^{b_2+k-1}}{(1+u/q)^{a_1+b_1+j}(1+u)^{a_2+b_2+k}}\,\mathrm{d}u \\
&= \frac{\exp(-\delta_1-\delta_2)}{\Gamma(a_1)\Gamma(a_2)} \frac{2^{a_1-b_2}}{(q+1)^{a_1+b_1}} \sum_{j=0}^\infty \sum_{k=0}^\infty \frac{\Gamma(a_1+b_1+j)\Gamma(a_2+b_2+k)\delta_1^j\delta_2^k}{\Gamma(b_1+j)\Gamma(b_2+k)j!k!} \\
&\quad \times \frac{1}{2^k(1+q)^j}\int_0^1 \frac{w^{b_2+k-1}(1-w)^{a_1+a_2-1}[1-(1-w)/q]^{b_1+j-1}}{[1-w/(q+1)]^{a_1+b_1+j}(1-w/2)^{a_2+b_2+k}}\,\mathrm{d}w.
\end{aligned}$$

In this case by writing $[1-(1-w)/q]^{b_1+j-1}$ as binomial sum and evaluating the resulting expression using (2.3) we obtain the p.d.f. $f_Q(q)$, for $q > 1$. ☐

Corollary 3.8. *Let X_1 and X_2 be independent random variables, $X_i \sim \mathrm{B3}(a_i, b_i)$, $i = 1, 2$. Further, assume that b_2 is a positive integer. Then, the p.d.f to $Q = X_1/X_2$ is given as*

$$\begin{aligned}
f_Q(q) &= \frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_2)} \frac{2^{a_2-b_1}q^{a_2-1}}{(1+q)^{a_2+b_2}} \sum_{r=0}^{b_2-1} \binom{b_2-1}{r} \frac{\Gamma(a_1+a_2+r)}{\Gamma(a_1+b_1+a_2+r)} q^r \\
&\quad \times F_1\left(b_1, a_1+b_1, a_2+b_2; a_1+b_1+a_2+r; \frac{1}{2}, \frac{q}{1+q}\right), \quad 0 < q < 1
\end{aligned}$$

and

$$\begin{aligned}
f_Q(q) &= \frac{\Gamma(a_1+b_1)\Gamma(a_2+b_2)}{\Gamma(a_1)\Gamma(a_2)\Gamma(b_1)} \frac{2^{a_1-b_2}}{(q+1)^{a_1+b_1}} \sum_{r=0}^{b_1+j-1} \binom{b_1-1}{r} \frac{\Gamma(a_1+a_2+r)}{\Gamma(a_1+a_2+b_2+r)} q^{-r} \\
&\quad \times F_1\left(b_2+r, a_1+b_1, a_2+b_2; a_1+a_2+b_2+r; \frac{1}{q+1}, \frac{1}{2}\right), \quad q > 1.
\end{aligned}$$

4 Conclusion

By using the traditional method of transformation of variables, we have obtained probability density functions of sum, difference, product and quotient of two independent random variables both having non-central beta type 3 distribution. These probability density functions have been expressed in series involving first hypergeometric function of Appell.

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Competing interests

The authors declare that they have no competing interests.

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