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Nonexistence of Radial Positive Solutions for A Quasilinear Elliptic Equations nonpositone Problems in an Annulus

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Research Article

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Abstract

In this paper, our main purpose is studying the nonexistence of radial positive solutions for the boundary-value problem:

$$\begin{cases} -\triangle_p u = \lambda f(u(x)), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega. \end{cases}$$

where $p > 1$,$\lambda > 0$, Ω is an annulus in $\mathbf{R}^N (N > 2)$ i.e. Ω=$\{x \in \mathbf{R}^N | R < |x| < \hat{R}\}(0 < R < \hat{R})$, f is a continuous nonlinear function and satisfies $f(0) < 0$ (the nonpositone case), f also has more than one zero.

Keywords: Nonpositone problem; radial positive solutions; nonexistence.
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1 Introduction

We consider the nonexistence of radial positive solutions of the problem

$$\begin{cases} -\triangle_p u = \lambda f(u(x)), & x \in \Omega; \\ u(x) = 0, & x \in \partial\Omega. \end{cases} \tag{1.1}$$

where $p > 1$, $\lambda > 0$, $\Omega \subset \mathbf{R}^N (N > 2)$ is an annulus i.e. Ω=$\{x \in \mathbf{R}^N | R < |x| < \hat{R}\}(0 < R < \hat{R})$; f has more than one zero and $f(0) < 0$ (the nonpositone case); $\triangle_p u =$ div$(|\nabla u|^{p-2}\nabla u)(p > 1)$.

In recent years, the existence, non-existence, asymptotic behavior and uniqueness of the positive solutions for the quasilinear eigenvalue problems (1.1) have been studied by many authors. See, for example [5-8, 11, 13, 15-19, 21-23, 27-33]. Chhetri and Girg [5] proved nonexistence results for (1.1) when Ω is a unit ball in $\mathbf{R}^N$ and f has only one zero. Rudd [6], nonexistence and existence results are proved when Ω is a connected and bounded subset of $\mathbf{R}^N$. Hai and Shivaji [7] studied elliptic systems related to (1.1) and proved the existence of positive solutions to (1.1) in some sublinear cases. The result of nontrivial solutions for p-Laplacian systems be proved by Hai and Wang [8]. When f is strictly

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increasing on $\mathbf{R}^+, f(0) = 0, \lim_{s\to 0^+} f(s)/s^{p-1} = 0$ and $f(s) \leq \alpha_1 + \alpha_2 s^\mu, 0 < \mu < p-1, \alpha_1, \alpha_2 > 0$, it was shown in Z.M. Guo [29] that there exist at least two positive solutions for Eqs (1.1) when λ is sufficiently large. If $\lim_{s\to 0^+} \inf f(s)/s^{p-1} > 0, f(0) = 0$ and the monotonicity hypothesis $(f(s)/s^{p-1})' < 0$ holds for all $s > 0$, it was proved by Guo and Webb [28] that the problem (1.1) has a unique positive solution when λ is sufficiently large. Moreover, it was also shown by Guo [30] that problem (1.1) has a unique positive large solution and at least one positive small solution when λ is large if f is nondecreasing; there exist $\alpha_1, \alpha_2 > 0$ such that $f(s) \leq \alpha_1 + \alpha_2 s^\beta, 0 < \beta < p-1; \lim_{s\to 0^+} \dfrac{f(s)}{s^{p-1}} = 0$, and there exist $T, Y > 0$ with $Y \geq T$ such that

$$(f(s)/s^{p-1})' > 0 \ \text{ for } \ s \in (0, T)$$

and

$$(f(s)/s^{p-1})' < 0 \ \text{ for } \ s > Y.$$

Recently, Hai [32] considered the case when Ω is an annular domains, and obtained the existence of positive large solutions for the problem (1.1) when λ sufficiently small. Xuan and Chen [33] proved the singular problem (1.1) has a unique positive radial solution if f is a continuous function and positive on $\overline{\Omega} = B_R$ (here B_R is a ball). The existence of entire solutions for (1.1) with singular and non-singular has been considered by Yang [26].

When $p = 2, f(0) < 0$, Ω is an annulus or a ball and f has more than one zero, the related results have been obtained by Hakimi and Zertiti [1] and Said and Zertiti [2]. When $p = 2, f(0) < 0$, f is a monotone nondecreasing nonlinearity and has only one zero, this problem has been studied by Castro and Shivaji [11] in the ball, and by Arcoya and Zertiti [4] in the annulus. The asymptotic behavior of positive solution have been obtained by Iaia [24-25]. In this paper, we further study this problem for Ω being an annulus and f has more than one zero. This extends and complements previous results in the literature of Hakimi and Zertiti [1]; Arcoya and Zertiti [4].

The paper is organized as follows. In section 2, we recall some lemmas that will be needed in the paper. In section 3, we give the main results and the proof of the main results in this paper.

2 Preliminaries

We consider radial solution to the quasilinear elliptic equation:

$$\begin{cases} -\text{div}(|\nabla u(x)|^{p-2}\nabla u(x)) = \lambda f(u(x)), & \text{x} \in \Omega, \\ u(x) = 0, & \text{x} \in \partial\Omega, \end{cases} \tag{2.1}$$

where $p > 1$, Ω is an annulus in $\mathbf{R}^N$ i.e. Ω=$\{x \in \mathbf{R}^N | R < |x| < \hat{R}\}(0 < R < \hat{R})$, $\lambda > 0$. While the function $f(u)$:$[0, +\infty) \to R$ satisfies the following assumptions:

(F1) $f \in C^1([0, +\infty), R)$, f has more than one zero and is not increasing entirely on $[0, +\infty)$;
(F2) $f(0) < 0$;
(F3) $\lim_{u\to+\infty} \frac{f(u)}{u^{p-1}} = +\infty$.

Remark 1. If f satisfies $(F1)$, in this paper we assume, without loss of generality, that f has three zeros $\beta_1 < \beta_2 < \beta_3$ and $f^{'}(\beta_i) \neq 0 (i \in \{1, 2, 3\})$. Then $f^{'} \geq 0$ on $[\beta_3, +\infty)$.

Then we state some preliminaries and three lemmas that will be needed later.

We observe that the nonexistence of radial positive solutions of (1.1) is equivalent to the nonexistence of positive solutions of the ordinary differential equation

$$\begin{cases} -(|u^{'}|^{p-2}u^{'})^{'} - \frac{N-1}{r}|u^{'}|^{p-2}u^{'} = \lambda f(u(r)), & r \in (R, \hat{R}), \\ u(R) = u(\hat{R}) = 0. \end{cases} \tag{2.2}$$

Or equivalently

$$\begin{cases} -(r^{N-1}|u^{'}|^{p-2}u^{'})^{'} = \lambda r^{N-1} f(u(r)), \quad r \in (R, \hat{R}), \\ u(R) = u(\hat{R}) = 0. \end{cases} \tag{2.2*}$$

Define $F(x) = \int_0^x f(t)dt$, from Remark 1, we know that f has three zeros, the number of zeros of F depends on f, then F has at most three zeros, denote by θ_i the zero of F, and let $\theta = \min\{\beta_2, \min\theta_i\}$.

By a modification of the method given in Hakimi and Zertiti [1]; Arcoya and Zertiti [4], we obtain the following lemmas.

Lemma 2.1. Assume (F_3) hold, if u_λ is a positive solution of (2.1), then for $\forall\lambda > 2$, there exists a positive number $M = M(r)$ (independent of λ) such that

$$u_\lambda \le M, r \in (R_0, \hat{R}],$$

where $R_0 = (R + \hat{R})/2$

Proof. If u_λ is a positive solution of (2.1), then

$$-(r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'})^{'} = \lambda r^{N-1} f(u_\lambda).$$

Multiplying the equation by u_λ, and integrating from R to $\hat{R}$, we obtain

$$-\int_R^{\hat{R}} (r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'})^{'} u_\lambda dr = \int_R^{\hat{R}} \lambda r^{N-1} f(u_\lambda) u_\lambda dr.$$

Hence

$$\int_R^{\hat{R}} r^{N-1}|u_\lambda^{'}|^p dr = \int_R^{\hat{R}} \lambda r^{N-1} f(u_\lambda) u_\lambda dr. \tag{2.3}$$

Let μ_1 is the first eigenvalue of the eigenvalue problem

$$-(r^{N-1}|v^{'}|^{p-2}v^{'})^{'} = \mu r^{N-1}|v|^{p-2}v, R < r < \hat{R}$$

$$v(R) = 0 = v(\hat{R}).$$

From (F_3) $\lim_{u\to+\infty} \frac{f(u)}{u^{p-1}} = +\infty$, there exists $\mu > \frac{\mu_1}{2}, c > 0$ such that

$$f(u) \ge \mu u^{p-1} + c. \tag{2.4}$$

From (2.3) and (2.4), we obtain

$$\int_R^{\hat{R}} r^{N-1}|u_\lambda^{'}|^p dr = \int_R^{\hat{R}} \lambda r^{N-1} f(u_\lambda) u_\lambda dr \ge \lambda\mu \int_R^{\hat{R}} r^{N-1} u_\lambda^p dr + c\lambda \int_R^{\hat{R}} r^{N-1} u_\lambda dr$$

$$c\lambda \int_R^{\hat{R}} r^{N-1} u_\lambda dr \le \int_R^{\hat{R}} r^{N-1}(|u_\lambda^{'}|^p - \lambda\mu u_\lambda^p) dr.$$

Because $\lambda > 2, \mu > \frac{\mu_1}{2}$, then

$$c\lambda \int_R^{\hat{R}} r^{N-1} u_\lambda dr \le \int_R^{\hat{R}} r^{N-1}(|u_\lambda^{'}|^p - \lambda\mu u_\lambda^p) dr < \int_R^{\hat{R}} r^{N-1}(|u_\lambda^{'}|^p - \mu_1 u_\lambda^p) dr. \tag{2.5}$$

On the other hand, let $\bar{r}_n = \max\{r \in (R, \hat{R}) : u_\lambda^{'}(r) = 0\}$, we remark that $\bar{r}_n \le R_0$, which was shown in A. Castro and R. Shivaji [11].

Since u_λ is non-increasing in $(R_0, \hat{R})$, then let $r \in (R_0, \hat{R})$, choosing $\delta > 0$ such that $R_0 < r - \delta$, we obtain

$$u_\lambda(r) \int_{r-\delta}^{r} t^{N-1} dt \leq \int_{r-\delta}^{r} t^{N-1} u_\lambda(t) dt$$

$$u_\lambda(r) \leq \frac{\int_{r-\delta}^{r} t^{N-1} u_\lambda(t) dt}{\int_{r-\delta}^{r} t^{N-1} dt} = \frac{\int_{r-\delta}^{r} t^{N-1} u_\lambda(t) dt}{\frac{1}{N}(r^N - (r-\delta)^N)} \leq \frac{\int_{R}^{\hat{R}} t^{N-1} u_\lambda(t) dt}{\frac{1}{N}(r^N - (r-\delta)^N)} \tag{2.6}$$

Let $\int_R^{\hat{R}} r^{N-1}(|u_\lambda^{'}|^p - \mu_1 u_\lambda^p) dr = k$. From (2.5), we have

$$\int_R^{\hat{R}} r^{N-1} u_\lambda(r) dr \leq \frac{k}{c\lambda} < \frac{k}{2c}.$$

This together with (2.6) imply

$$u_\lambda(r) \leq \frac{\int_R^{\hat{R}} t^{N-1} u_\lambda(t) dt}{\frac{1}{N}(r^N - (r-\delta)^N)} \leq \frac{Nk}{2c(r^N - (r-\delta)^N)} = M. \tag{2.7}$$

The lemma is proved.

Lemma 2.2. Assume (F_1)-(F_3) and let $R_1 \in (R_0, \hat{R})$, $c \in (\beta_1, \theta)$, then there exists $\lambda_1 > 0$ and $r_1 = r_1(\lambda) \in (R_0, R_1)$, such that when $\lambda > \lambda_1$ all positive solution u_λ of (2.2) satisfying $u_\lambda(r_1) < c$.

Proof. By contradiction, suppose that there exists a sequence $\{\lambda_n\} \subset (0, +\infty)$ and $\lim_{n\to\infty} \lambda_n = +\infty$ such that

$$u_{\lambda_n}(r) \geq c, \tag{2.8}$$

for $\forall r \in (R_0, R_1], \forall n \in N$.

Let $\bar{r}_n = \max\{r \in (R, \hat{R}) : u_{\lambda_n}^{'}(r) = 0\}$, then $u_{\lambda_n}^{'}(r) < 0$, $u_{\lambda_n}(r) \leq u_{\lambda_n}(\bar{r}_n)$, for $\forall r \in (\bar{r}_n, \hat{R})$. Also, we remark that $\bar{r}_n \leq R_0$, was shown in A. Castro and R. Shivaji [11].

It follows that $u_{\lambda_n}^{'}(\bar{r}_n) = 0$ and $u_{\lambda_n}^{''}(\bar{r}_n) \leq 0$, then from (2.2)

$$-(|u_{\lambda_n}^{'}(\bar{r}_n)|^{p-2} u_{\lambda_n}^{'}(\bar{r}_n))^{'} - \frac{N-1}{r} |u_{\lambda_n}^{'}(\bar{r}_n)|^{p-2} u_{\lambda_n}^{'}(\bar{r}_n) = \lambda_n f(u_{\lambda_n}(\bar{r}_n)),$$

which implies $f(u_{\lambda_n}(\bar{r}_n)) = 0$, hence $u_{\lambda_n}(\bar{r}_n) = \beta_1, \beta_2$ or β_3.

Case 1. $u_{\lambda_n}(\bar{r}_n) = \beta_1$. From $c \in (\beta_1, \theta)$, we know that $u_{\lambda_n} = \beta_1 < c$, which contradicts with the assuming (2.8) that $u_{\lambda_n}(r) \geq c$.

Case 2. $u_{\lambda_n}(\bar{r}_n) = \beta_2$. Consider the following two sets:

$$\Phi_n = \{r \in [R_1, \hat{R}] : \beta_1 \leq u_{\lambda_n}(r) \leq \frac{3\beta_1 + c}{4}\},$$

$$\Psi_n = \{r \in [R_1, \hat{R}] : \frac{2(\beta_1 + c)}{4} \leq u_{\lambda_n}(r) \leq \frac{\beta_1 + 3c}{4}\}.$$

Since $(\beta_1, \frac{3\beta_1+c}{4}), (\frac{2(\beta_1+c)}{4}, \frac{\beta_1+3c}{4}) \subset u_{\lambda_n}((R_1, \hat{R}))$, by the intermediate value theorem, Φ_n and Ψ_n are not empty. Then let

$$\underline{a}(n) = \inf_r \Psi_n, \ \ \overline{a}(n) = \sup_r \Psi_n; \ \underline{b}(n) = \inf_r \Phi_n, \ \ \overline{b}(n) = \sup_r \Phi_n.$$

It follows that $\underline{a}(n) \leq \overline{a}(n) \leq \underline{b}(n) \leq \overline{b}(n)$. Take $r_0 \in [\underline{a}(n), \overline{b}(n)]$ and taking into account that

$$-(r^{N-1}|u_{\lambda_n}^{'}|^{p-2} u_{\lambda_n}^{'})^{'} = \lambda_n r^{N-1} f(u_{\lambda_n})$$

integrating on $[\bar{r}_n, r_0]$, we have

$$-\int_{\bar{r}_n}^{r_0}(s^{N-1}|u'_{\lambda_n}|^{p-2}u'_{\lambda_n})'ds = \lambda_n\int_{\bar{r}_n}^{r_0}s^{N-1}f(u_{\lambda_n}(s))ds$$

because $u'_{\lambda_n}(\bar{r}_n) = 0$, we deduce

$$\begin{aligned} -r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0) &= \lambda_n\int_{\bar{r}_n}^{r_0}s^{N-1}f(u_{\lambda_n}(s))ds \\ &\geq \lambda_n R^{N-1}\int_{R_0}^{r_0}f(u_{\lambda_n}(s))ds \\ &\geq \lambda_n R^{N-1}\int_{u_{\lambda_n}(R_0)}^{u_{\lambda_n}(r_0)}\frac{f(t)}{u'_{\lambda_n}(u_{\lambda_n}^{-1}(t))}dt \\ &= \lambda_n R^{N-1}\int_{u_{\lambda_n}(r_0)}^{u_{\lambda_n}(R_0)}\frac{f(t)}{-u'_{\lambda_n}(u_{\lambda_n}^{-1}(t))}dt \end{aligned}$$

for $t = u_{\lambda_n}(s), ds = \frac{dt}{u'_{\lambda_n}(u_{\lambda_n}^{-1}(t))}$.

Hence we obtain

$$-r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0) \geq \lambda_n R^{N-1}\int_{u_{\lambda_n}(r_0)}^{u_{\lambda_n}(R_0)}\frac{f(t)}{-u'_{\lambda_n}(u_{\lambda_n}^{-1}(t))}dt \tag{2.9}$$

Let $u'_{\lambda_n}(s_0) = \inf_{[R_0,r_0]} u'_{\lambda_n}(s)$, we have

$$\begin{aligned} -r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0)(-u'_{\lambda_n}(s_0)) &\geq \lambda_n R^{N-1}\int_{u_{\lambda_n}(r_0)}^{u_{\lambda_n}(R_0)}f(t)dt \\ &\geq \lambda_n R^{N-1}\int_{\frac{\beta_1+3c}{4}}^{c}f(t)dt \qquad (2.10) \end{aligned}$$

On the other hand, for all $r \in (\underline{a}(n), \bar{b}(n))$, $u_{\lambda_n}(r) \in (\beta_1, \frac{\beta_1+3c}{4}) \subset (\beta_1, \beta_2)$, which implies

$$-(r^{N-1}|u'_{\lambda_n}(r)|^{p-2}u'_{\lambda_n}(r))' = \lambda_n r^{N-1}f(u_{\lambda_n}(r)) > 0, \quad r \in (\underline{a}(n), \bar{b}(n))$$

so, $r \mapsto -r^{N-1}|u'_{\lambda_n}(r)|^{p-2}u'_{\lambda_n}(r)$ is increasing on $(\underline{a}(n), \bar{b}(n))$, then for all $s_0 \in [R_0, r_0]$, we have

$$\begin{aligned} -s_0^{N-1}|u'_{\lambda_n}(s_0)|^{p-2}u'_{\lambda_n}(s_0) &\leq -r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0) \\ -u'_{\lambda_n}(s_0) &\leq \frac{-r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0)}{s_0^{N-1}|u'_{\lambda_n}(s_0)|^{p-2}} \end{aligned}$$

That is

$$-r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0)(-u'_{\lambda_n}(s_0)) \leq \frac{(-r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0))^2}{s_0^{N-1}|u'_{\lambda_n}(s_0)|^{p-2}}. \tag{2.11}$$

Using (2.10)(2.11), we get

$$\lambda_n R^{N-1}\int_{\frac{\beta_1+3c}{4}}^{c}f(t)dt \leq \frac{(-r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0))^2}{s_0^{N-1}|u'_{\lambda_n}(s_0)|^{p-2}} = \frac{r_0^{2(N-2)}|u'_{\lambda_n}(r_0)|^{2(p-2)+2}}{s_0^{N-1}|u'_{\lambda_n}(s_0)|^{p-2}}.$$

Therefore
$$\lim_{n\to+\infty} u'_{\lambda_n}(r_0) = -\infty. \tag{2.12}$$

Now, Let $r_1 \in [\underline{a}(n), \overline{a}(n)]$ and $r_2 \in [\underline{b}(n), \overline{b}(n)]$, by the mean value theorem, there exists $r^* \in (r_1, r_2)$ such that

$$\begin{aligned} u_{\lambda_n}(r_2) &= u_{\lambda_n}(r_1) + u'_{\lambda_n}(r^*)(r_2 - r_1) \\ &\le u_{\lambda_n}(R_1) + (\underline{b}(n) - \overline{a}(n))u'_{\lambda_n}(r^*) \\ &\le u_{\lambda_n}(R_1) + \inf_n(\underline{b}(n) - \overline{a}(n))u'_{\lambda_n}(r^*) \end{aligned}$$

From Lemma 2.2, $u_{\lambda_n}(R_1) \le M$, for all n and some $M = M(R_1) > 0$. Moreover, from (2.12), $\lim_{n\to+\infty} u'_{\lambda_n}(r^*) = -\infty$ and $\inf_n(\underline{b}(n) - \overline{a}(n)) > 0$, then we have

$$\lim_{n\to+\infty} u'_{\lambda_n}(r_2) = -\infty$$

which contradicts with $u_{\lambda_n}(r) \ge c$, for all $n \in N$ (by (2.8)).

Case 3. $u_{\lambda_n}(\bar{r}_n) = \beta_3$. Let $r_{\beta_2} = \max\{r_n \in (R, \hat{R}) : u_{\lambda_n}(r_n) = \beta_2\}$, take $r_0 \in [\underline{a}(n), \overline{b}(n)]$, for

$$-(r^{N-1}|u'_{\lambda_n}(r)|^{p-2}u'_{\lambda_n}(r))' = \lambda_n r^{N-1} f(u_{\lambda_n}(r)).$$

integrate it from $\bar{r}_n$ to r_0, we get

$$\begin{aligned} -r_0^{N-1}|u'_{\lambda_n}(r_0)|^{p-2}u'_{\lambda_n}(r_0) &= \lambda_n \int_{\bar{r}_n}^{r_0} s^{N-1} f(u_{\lambda_n}(s))ds \\ &= \lambda_n [\int_{\bar{r}_n}^{r_{\beta_2}} s^{N-1} f(u_{\lambda_n}(s))ds + \int_{r_{\beta_2}}^{r_0} s^{N-1} f(u_{\lambda_n}(s))ds] \\ &\ge \lambda_n \int_{r_{\beta_2}}^{r_0} s^{N-1} f(u_{\lambda_n}(s))ds. \end{aligned}$$

Since for all $s \in [\bar{r}_n, r_{\beta_2}]$, $u_{\lambda_n}(s) \in [\beta_2, \beta_3]$, then $f(u_{\lambda_n}(s)) \ge 0$ (by (F_1) and Remark 1)

Then as in the case 2, we obtain a contradiction with the positivity of u_{λ_n}. Thus, combining case 1, 2 and 3, the lemma is proved.

Lemma 2.3. Assume (F_2) is satisfied, let $R_2 \in (R_0, \hat{R})$ and $\bar{c} > 1$, then there exists $\lambda_2 > 0$ such that for all $\lambda \ge \lambda_2$, every positive solution u_λ satisfies $\frac{\beta_1}{\bar{c}} \in u_\lambda([R_2, \hat{R}])$.

Proof. Define $b_\lambda = \max\{r \in (R, \hat{R}) : u_\lambda(r) = \frac{\beta_1}{\bar{c}}\}$. Now, we will prove that $\lim_{\lambda\to+\infty} b_\lambda = \hat{R}$.
From (2.2^*), we have

$$-(r^{N-1}|u'_\lambda(r)|^{p-2}u'_\lambda(r))' = \lambda r^{N-1} f(u_\lambda(r)),$$

integrate the equation from b_λ to $\hat{R}$

$$\int_{b_\lambda}^{\hat{R}} (r^{N-1}|u'_\lambda(r)|^{p-2}u'_\lambda(r))' dr = \int_{b_\lambda}^{\hat{R}} (-\lambda r^{N-1} f(u_\lambda(r)))dr \ge \int_{b_\lambda}^{\hat{R}} \lambda r^{N-1} K dr$$

(with the fact that $u_\lambda(r) < \frac{\beta_1}{\bar{c}}$ for all $r \in (b_\lambda, \hat{R}]$ and the definition of $K = -\max\{f(s) : s \in [0, \frac{\beta_1}{\bar{c}}]\} > 0$). Then

$$\hat{R}^{N-1}|u'_\lambda(\hat{R})|^{p-2}u'_\lambda(\hat{R}) - b_\lambda^{N-1}|u'_\lambda(b_\lambda)|^{p-2}u'_\lambda(b_\lambda) \ge \frac{\lambda}{N} K(\hat{R}^N - b_\lambda^N) > 0. \tag{2.13}$$

On the other hand, multiplying equation (2.2^*) by $r^{N-1}|u_\lambda^{'}(r)|^{p-2}u_\lambda^{'}(r)$ and integrating from b_λ to $\hat{R}$, we have

$$\begin{aligned} -\int_{b_\lambda}^{\hat{R}}(r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'})^{'}(r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'})dr &= \lambda\int_{b_\lambda}^{\hat{R}} r^{2(N-1)}f(u_\lambda)|u_\lambda^{'}|^{p-2}u_\lambda^{'}dr \\ &= \lambda\int_{b_\lambda}^{\hat{R}} r^{2(N-1)}[F(u_\lambda)]^{'}|u_\lambda^{'}|^{p-2}dr \end{aligned} \tag{2.14}$$

Integrating twice by parts, we have

$$\begin{aligned} &-\int_{b_\lambda}^{\hat{R}}(r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'})^{'}(r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'})dr \\ &= -\int_{b_\lambda}^{\hat{R}}(r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'})d(r^{N-1}|u_\lambda^{'}|^{p-2}u_\lambda^{'}) = -\frac{1}{2}(r^{N-1}|u_\lambda^{'}(r)|^{p-2}u_\lambda^{'}(r))^2|_{b_\lambda}^{\hat{R}} \\ &= -\frac{1}{2}\{\hat{R}^{2(N-1)}|u_\lambda^{'}(\hat{R})|^{2(p-2)}u_\lambda^{'}(\hat{R})^2 - b_\lambda^{2(N-1)}|u_\lambda^{'}(b_\lambda)|^{2(p-2)}u_\lambda^{'}(b_\lambda)^2\}. \end{aligned} \tag{2.15}$$

On the other hand

$$\begin{aligned} &\lambda\int_{b_\lambda}^{\hat{R}} r^{2(N-1)}[F(u_\lambda)]^{'}|u_\lambda^{'}|^{p-2}dr \\ &= \lambda\hat{R}^{2(N-1)}F(u_\lambda(\hat{R}))|u_\lambda^{'}(\hat{R})|^{p-2} - \lambda b_\lambda^{2(N-1)}F(u_\lambda(b_\lambda))|u_\lambda^{'}(b_\lambda)|^{p-2} - \lambda\int_{b_\lambda}^{\hat{R}}F(u_\lambda)d(r^{2(N-1)}|u_\lambda^{'}|^{p-2}) \\ &= -\lambda b_\lambda^{2(N-1)}F(\frac{\beta_1}{\bar{c}})|u_\lambda^{'}(b_\lambda)|^{p-2} - \lambda\int_{b_\lambda}^{\hat{R}}F(u_\lambda)d(r^{2(N-1)}|u_\lambda^{'}|^{p-2}) \end{aligned} \tag{2.16}$$

(because $u_\lambda(\hat{R}) = 0, u_\lambda(b_\lambda) = \frac{\beta_1}{\bar{c}}$).

Since $u_\lambda(r) \in (0, \frac{\beta_1}{\bar{c}})$, for all $r \in (b_\lambda, \hat{R})$ and F is decreasing in $(0, \beta_1)$, by (2.14)-(2.16), we have

$$\begin{aligned} &\frac{1}{2}\{b_\lambda^{2(N-1)}|u_\lambda^{'}(b_\lambda)|^{2(p-2)}u_\lambda^{'}(b_\lambda)^2 - \hat{R}^{2(N-1)}|u_\lambda^{'}(\hat{R})|^{2(p-2)}u_\lambda^{'}(\hat{R})^2\} \\ &= -\lambda b_\lambda^{2(N-1)}F(\frac{\beta_1}{\bar{c}})|u_\lambda^{'}(b_\lambda)|^{p-2} - \lambda\int_{b_\lambda}^{\hat{R}}F(u_\lambda)d(r^{2(N-1)}|u_\lambda^{'}|^{p-2}) \\ &\le -\lambda b_\lambda^{2(N-1)}F(\frac{\beta_1}{\bar{c}})|u_\lambda^{'}(b_\lambda)|^{p-2} - \lambda F(\frac{\beta_1}{\bar{c}})\int_{b_\lambda}^{\hat{R}}d(r^{2(N-1)}|u_\lambda^{'}|^{p-2}) \\ &= -\lambda b_\lambda^{2(N-1)}F(\frac{\beta_1}{\bar{c}})|u_\lambda^{'}(b_\lambda)|^{p-2} - \lambda F(\frac{\beta_1}{\bar{c}})\{\hat{R}^{2(N-1)}|u_\lambda^{'}(\hat{R})|^{p-2} - b_\lambda^{2(N-1)}|u_\lambda^{'}(b_\lambda)|^{p-2}\} \\ &= -\lambda F(\frac{\beta_1}{\bar{c}})\hat{R}^{2(N-1)}|u_\lambda^{'}(\hat{R})|^{p-2} \end{aligned} \tag{2.17}$$

Because $A - B \le \sqrt{A^2 - B^2}$ for all $A \ge B \ge 0$, then, since $u_\lambda^{'}(b_\lambda) < 0$ by definition of b_λ and $u^{'}(\hat{R}) \le 0$ by X. Garaizar [9], we define

$$A = -b_\lambda^{N-1}u^{'}(b_\lambda) > 0, \quad B = -\hat{R}_\lambda^{N-1}u^{'}(\hat{R}) > 0,$$

by (2.13) it follows that $A > B > 0$, from (2.13) and (2.17) we deduce

$$\begin{aligned}
\frac{\lambda}{N}K(\hat{R}^N - b_\lambda^N) &\leq -b_\lambda^{N-1}|u_\lambda^{'}(b_\lambda)|^{p-2}u_\lambda^{'}(b_\lambda) - (-\hat{R}^{N-1}|u_\lambda^{'}(\hat{R})|^{p-2}u_\lambda^{'}(\hat{R})) \\
&\leq \sqrt{b_\lambda^{2(N-1)}|u_\lambda^{'}(b_\lambda)|^{2(p-2)}u_\lambda^{'}(b_\lambda)^2 - \hat{R}^{2(N-1)}|u_\lambda^{'}(\hat{R})|^{2(p-2)}u_\lambda^{'}(\hat{R})^2} \\
&\leq \sqrt{-2\lambda F(\frac{\beta_1}{\bar{c}})\hat{R}^{2(N-1)}|u_\lambda^{'}(\hat{R})|^{p-2}}
\end{aligned}$$

Then we have

$$\hat{R}^N - b_\lambda^N \leq \frac{\sqrt{2}N\hat{R}^{N-1}\sqrt{|u_\lambda^{'}(\hat{R})|^{p-2}}\sqrt{-F(\frac{\beta_1}{c})}}{K\sqrt{\lambda}}$$

hence the limit follows to be equal $\lim_{\lambda\to+\infty} b_\lambda = \hat{R}$, thus the Lemma is proved.

3 Main result

By a modification of the method given in (S. Hakimi and A. Zertiti [1]; D. Arcoya and A. Zertiti [4]), we obtain the following results.

Theorem 3.1. Under the assuming $(F_1) - (F_3)$, there exists a positive real number λ_0 such that if $\lambda > \lambda_0$, problem (1.1) has no radial positive solution.

Proof. Let $c \in (\beta_1, \theta)$,$\bar{c} > 1$ and $R_1, R_2 \in (R_0, \hat{R})$ such that $R_1 < R_2$. Consider λ_1, λ_2 given respectively by Lemma 2.2 and Lemma 2.3, and choose $\lambda^* \geq \{\lambda_1, \lambda_2\}$ such that

$$\lambda^* L + \frac{(p-1)u^p}{p} < 0. \tag{3.1}$$

where $L = \max\{F(s) : \frac{\beta_1}{\bar{c}} \leq s \leq c\}$. Then problem (1.1) has no radial positive solution.

By contradiction, assume that there exists $\lambda \geq \lambda^*$ such that problem (1.1) has at least one positive solution u_λ.

Because $\lambda \geq \lambda^* \geq \lambda_i (i = 1, 2)$, from Lemmas 2.2 and 2.3, we deduce there exist $t_1 \in (R_0, R_1], t_2 \in [R_2, \hat{R}]$ satisfying $u_\lambda(t_1) < c$ and $u_\lambda(t_2) = \frac{\beta_1}{\bar{c}}$

Then by the mean value theorem there exists $t_3 \in [t_1, t_2]$ such that

$$|u_\lambda^{'}(t_3)| = \frac{|u_\lambda(t_2) - u_\lambda(t_1)|}{t_2 - t_1} \leq \frac{|c + \frac{\beta_1}{\bar{c}}|}{R_2 - R_1} = \mu$$

Since $u_\lambda(t_3) \in [\frac{\beta_1}{\bar{c}}, c]$, it follows that $F(u_\lambda(t_3)) \leq L < 0$. Thus consider the energy function $E(r) = \lambda F(u_\lambda(r)) + \frac{p-1}{p}|u_\lambda^{'}(r)|^p$, for all $\lambda \geq \lambda^*$,

$$E(t_3) = \lambda F(u_\lambda(t_3)) + \frac{p-1}{p}|u_\lambda^{'}(t_3)|^p \leq \lambda L + \frac{p-1}{p}\mu^p \leq \lambda^* L + \frac{p-1}{p}\mu^p < 0 \tag{3.2}$$

On the other hand

$$
\begin{aligned}
E^{'}(r) &= \lambda F^{'}(u_\lambda(r))u_\lambda^{'}(r) + (\frac{p-1}{p}|u_\lambda^{'}(r)|^p)^{'} \\
&= \lambda f(u_\lambda(r))u_\lambda^{'}(r) + (p-1)|u_\lambda^{'}(r)|^{p-2}u_\lambda^{'}(r)u_\lambda^{''}(r) \\
&= -(|u_\lambda^{'}(r)|^{p-2}u_\lambda^{'}(r))^{'}u_\lambda^{'}(r) - \frac{N-1}{r}(|u_\lambda^{'}(r)|^p) + (p-1)|u_\lambda^{'}(r)|^{p-2}u_\lambda^{'}(r)u_\lambda^{''}(r) \\
&= -(p-1)|u_\lambda^{'}(r)|^{p-2}u_\lambda^{'}(r)u_\lambda^{''}(r) - \frac{N-1}{r}(|u_\lambda^{'}(r)|^p) + (p-1)|u_\lambda^{'}(r)|^{p-2}u_\lambda^{'}(r)u_\lambda^{''}(r) \\
&= -\frac{N-1}{r}(|u_\lambda^{'}(r)|^p) \le 0
\end{aligned}
$$

So $E(r)$ is a nonincreasing function,

$$E(t_3) > E(\hat{R}) = \lambda F(u_\lambda(\hat{R})) + \frac{p-1}{p}|u_\lambda^{'}(\hat{R})|^p = \frac{p-1}{p}|u_\lambda^{'}(\hat{R})|^p \ge 0,$$

which contradicts (3.2). Hence the theorem is proved.

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Competing interests

The authors declare that they have no competing interests.

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