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Characterization of Delta Operator for Appell Polynomials

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Authors' contributions

This work was carried out in collaboration between both authors. Author AM designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript and managed literature searches. Author CE managed the analyses of the study and literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper, we introduce a new approach for the Appell polynomials via the sequential representation of the delta operator. Moreover, a theorem which gives the necessary condition for Appell polynomials is proposed. The main objective of this paper is to investigate the characterization of the delta operator for the Bernoulli, the Hermite and the Genocchi polynomials. From our investigation, we are able to prove many interesting propositions for the above mentioned.

Keywords: Delta operator; sheffer polynomials; appell polynomials.

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1 Introduction

The use of Special Polynomials in Mathematics is very impressive: they are important in combinatorics, number theory, numerical analysis, operator theory, stochastic processes etc. In particular, some properties of Sheffer polynomials namely Bernoulli, Genocchi and Euler are discussed in [1].

Octavian Agratini [2] derived an identity involving the delta operator and basic polynomial sequence. G.C.Rota [3] contains a detailed study of delta operator, basic polynomial sequence, Sheffer polynomials and Appel polynomials. In [4], The delta operator is uniquely fixed by a sequence of real numbers. A theorem from G.C.Rota which gives the necessary and sufficient conditions for the basic polynomials related to some delta operator is reconstructed in terms of three new identities in [5]. A new definition of the inverse operator of any operator which applies a space of differentiable functions onto itself is proposed in [6].

The aim of the present paper is to propose some results tied to the Appell polynomials corresponding to the delta operator. The rest of the paper is organized in three sections. The second section summarizes some known definitions and theorems from G.C.Rota [3]. In this third section, we obtain two theorems. The first theorem deals with the delta operator when Q is a constant multiple of the usual derivative D. The second theorem discuss about the sequential representation of the delta operator. In the fourth section, we proposed a theorem which gives the necessary condition for Appell polynomials and this theorem is verified by some popular Appell polynomials.

2 Preliminaries

Operational Calculus is a technique used to reduce the differential problems into algebracic problems. This technique is fully utilized by Oliver Heaviside in 1893. A different approach to Operational Calculus was developed in 1930 by Jan Mikusinski and a rebirth is given in 1970's by G.C.Rota. In this section, we recall terminology, notation, some basic definitions and results of the finite operator calculus, as it has been introduced by G.C. Rota. The proofs of several results are skipped. But they are easily read from the reference G.C.Rota [3].

Let F be a Field of characteristic zero, preferably the real number field. Let $p(x)$ be a polynomial in one variable defined over F. A sequence of polynomials is $\{p_n(x)/n \in \mathbb{Z}^+ \cup \{0\}\}$, where $p_n(x)$ is exactly of degree n.

Definition 1.

(a). An operator E^a is said to be a shift operator if $E^a p(x) = p(x+a)$, for all polynomials $p(x)$ in one variable defined over the field F and $a \in F$.

(b). A linear operator T which commutes with all shift operators is called a shift invariant. In symbols, $TE^a = E^a T, \ \ \forall a \in F$.

(c). A delta operator usually denoted by the letter Q, is a shift-invariant operator for which Qx is a non zero constant.

Thus every delta operator Q is shift invariant. But a shift invariant operator need not be a delta operator.

Theorem 1.

(a). If Q is a delta operator, then $Qa = 0$ for every constant $'a'$.

(b). If $p(x)$ is a polynomial of degree n, then $Qp(x)$ is a polynomial of degree $n-1$.

The delta operators possesses many of the properties of the usual derivative D. The above theorems (a) and (b) in Theorem (1) are good examples.

Definition 2.
Let Q be a delta operator, A polynomial sequence $p_n(x)$ is called the sequence of basic polynomials for Q if :
i). $p_0(x) = 1$ ii). $p_n(0) = 0$, whenever $n > 0$ iii). $Qp_n(x) = np_{n-1}(x)$

The basic polynomials are a large class of polynomial sequences that include the Monomials $\{x^n; n = 0, 1, 2, \cdots\}$, the sequences of Lower factorials $[x]_n$, Upper factorials $[x]^n$, the Abel polynomials and many others.

Theorem 2.
Every delta operator has a unique sequence of basic polynomials.

Definition 3.
A polynomial sequence $s_n(x)$ is called a Sheffer set or a set of Sheffer polynomials for the delta operator Q if

1. $s_0(x) = c \neq 0,$
2. $Qs_n(x) = ns_{n-1}(x)$

Sheffer polynomials are a set of polynomial sequences that include Abel polynomials, Laguerre polynomials, Meixner polynomials, Poisson-Charlier polynomials, Bell polynomials and many others.

Definition 4.
A sequence of polynomials $p_n(x)$, $n = 0, 1, 2, \cdots$ of exact degree n is called an Appell sequence with respect to the derivative D if

1. $p_0(x) = c \neq 0,$
2. $\frac{d}{dx}\, p_n(x) = np_{n-1}$

The generating function of the Appell polynomials of the form

$$p(t)\; e^{xt} = \sum_{n=0}^{\infty} p_n(x)\; \frac{t^n}{n!}$$

Where $p(t)$ has expansion

$$p(t) = \sum_{n=0}^{\infty} p_n\; \frac{t^n}{n!} \qquad (p_0 \neq 0)$$

Some known Appell polynomials are the Bernoulli polynomials, the Hermite polynomials, the Genocchi polynomials, Gold-Hopper polynomials and many others.

For some sequence $\{c_n\}, n = 0, 1, 2, \cdots$, of scalars with $c_0 \neq 0$,

$$p_n(x) = \Big(\sum_{k=0}^{\infty} \frac{c_k}{k!}\; D^k\Big)\; x^n, \quad where\ D = \frac{d}{dx}$$

Also, for $n = 0, 1, 2, \cdots,$

$$p_n(x+y) = \sum_{k=0}^{n} \binom{n}{k}\; p_k(x)\; p_{n-k}(y)$$

The second condition in Definition (3) is the essential one for analysing the delta operator for

Sheffer set. If Q is replaced by the usual derivative D, then it becomes an Appell polynomials. The fourth section is a detailed study of the sequences of Appell polynomials and their characterization of delta operators. Now we attempt to formulate this delta operator Q in terms of a sequence of real numbers in the next section.

3 Sequential Representation of Delta Operator Q

According to G.C.Rota[3], Delta operator Q possesses many of the properties of usual derivative operator D. Generally usual derivative D is a delta operator. But the converse is not true.

Suppose if we define the delta operator by $Q(x^n) = x^{n-1}$, $n \in \mathbb{Z}^+ \cup \{0\}$, then $E^aQ(x_n) \neq QE^a(x^n)$. It means that Q is not shift invariant. Hence we conclude that $Q(x^n) = x^{n-1}$, $n \in \mathbb{Z}^+ \cup \{0\}$ will not be a delta operator.

By Theorem 1(b), if we Take $Q(x^n) = A_n\ x^{n-1}$ where A_n is a real constant , then it leads to the following theorem

Theorem 3.
If Q is a delta operator and $Q(x^n) = A_n\ x^{n-1}$, where A_n is a real constant , $n \in \mathbb{Z}^+$, then $Q = kD$ where k is a real constant and D is the usual derivative.

Proof :
Since Q is Shift invariant we have

$$E^aQ(x^n) = QE^a(x^n) \tag{3.1}$$

Putting $Q(x^n) = A_n\ x^{n-1}$ in left hand side

$$\begin{aligned} E^a\ Q(x^n) &= E^a\ (A_n\ x^{n-1}) = A_n\ E^a\ x^{n-1} = A_n\ (x+a)^{n-1} \\ &= A_n\ [x^{n-1} + \binom{n-1}{1} x^{n-2}a + \binom{n-1}{2} x^{n-3}a^2 + \cdots + a^{n-1}] \end{aligned}$$

Therefore,

$$E^a\ Q(x^n) = A_n\ x^{n-1} + A_n \binom{n-1}{1} x^{n-2}a + A_n \binom{n-1}{2} x^{n-3}a^2 + \cdots + A_n a^{n-1} \tag{3.2}$$

and also,

$$\begin{aligned} QE^a(x^n) &= Q(x+a)^n \\ &= Q[(x^n + \binom{n}{1} x^{n-1}a + \binom{n}{2} x^{n-2}a^2 + \cdots + a^n] \\ &= Q(x^n) + a\binom{n}{1} Q(x^{n-1}) + a^2 \binom{n}{2} Q(x^{n-2}) + \cdots + a^{n-1}Q(x) + Q(a^n) \end{aligned}$$

Since $Q(x^n) = A_n\ x^{n-1}$ and $Q(a^n) = 0$, the right hand side of Equation (3.1) becomes

$$QE^a(x^n) = A_n\ x^{n-1} + a\binom{n}{1} A_{n-1}\ x^{n-2} + a^2 \binom{n}{2} A_{n-2}\ x^{n-3} + \cdots + a^{n-1}A_1 \tag{3.3}$$

Equating the corresponding terms from equations (3.2) and (3.3), we get

$$A_n \binom{n-1}{r} = \binom{n}{r} A_{n-r}, \quad r = 1, 2, 3, \cdots n$$

and hence $A_n = nA_1$.

Therefore,

$$Q(x^n) = nA_1 \; x^{n-1}$$

By taking $A_1 = k$, a real constant , we get

$$Q(x^n) = k \; n \; x^{n-1}$$

That is,

$$Q = kD \quad \Box$$

From the above Theorem (3), we observe that the delta operator reduces to a constant multiple of usual derivative D. Now we attempt to formulate $Q(x^n)$ in terms of a sequence of real numbers. By Theorem (1), we obtain the following Theorem.

Theorem 4.
For the monomials $\{x^n : \; n \in \mathbb{Z}^+ \cup \{0\}\}$, and α_r an arbitary real numbers,

$$Q(x^n) = \sum_{r=1}^{n} \binom{n}{r} \; \alpha_r \; x^{n-r}. \tag{3.4}$$

Proof .
Taking $Q(x) = \alpha_1 \neq 0$ and construct $Q(x^2) = c_0 \; x + c_1$. Since Q is shift invariant, we have $E^a Q(x^2) = QE^a(x^2)$. Solving we get $c_0 = 2\alpha_1$ and c_1 is a new independent constant which may be taken as α_2. Hence $Q(x^2) = 2\alpha_1 x + \alpha_2$. Thus the theorem is true for $n = 1$ and 2.

Let us assume that the result is true for $n = k$.

Therefore ,

$$Q(x^k) = \sum_{r=1}^{k} \binom{k}{r} \; \alpha_r \; x^{k-r} = \binom{k}{1} \; \alpha_1 \; x^{k-1} + \binom{k}{2} \; \alpha_2 \; x^{k-2} + \cdots + \binom{k}{r} \; \alpha_r \; x^{k-r} + \cdots + \alpha_k \tag{3.5}$$

Since $\{x^n\}$ is a basic polynomial sequence, it satisfies $Qp_n(x) = np_{n-1}(x)$ and hence we have,

$$Q(x^k) = k \; x^{k-1} \tag{3.6}$$

From Equation (3.6), we see that the delta operator Q is a usual derivative D.

From Equations (3.5) and (3.6) ,

$$\binom{k}{1} \; \alpha_1 \; x^{k-1} + \binom{k}{2} \; \alpha_2 \; x^{k-2} + \cdots + \binom{k}{r} \; \alpha_r \; x^{k-r} + \cdots + \alpha_k = k \; x^{k-1} \tag{3.7}$$

By comparing the corresponding terms, we have $\alpha_1 = 1$ and $\alpha_j = 0, \; j = 2, 3, \cdots k$

Therefore, the result is true for $n = k$ means that

$$\alpha_1 = 1 \; and \; \alpha_j = 0 \; (j = 2, 3, \cdots k) \tag{3.8}$$

Now we have to show that this result is true for $n = k + 1$

$$Q(x^{k+1}) = Q(x^k \; x) = Q(x^k) \; x + Q(x) \; x^k \; = (k+1) \; x^k$$

Thus we have

$$Q(x^{k+1}) = (k+1) \; x^k \tag{3.9}$$

On other hand, using the property that $Qp_n(x) = n\ p_{n-1}(x)$, we have

$$Q(x^{k+1}) = (k+1)\ p_k(x) = (k+1)\ x^k \tag{3.10}$$

From the Equations (3.9) and (3.10), we conclude that the result is true for all $n = k + 1$

Thus we proved the Theorem 4. □

Here, $Q(x^n)$ has n independent parameters, $\alpha_i, (i = 1, 2, 3 \ldots n)$. These parameters are unique. Hence we conclude that any delta operator may be fixed uniquely by Equation (3.4). To study the delta operator, we need analyse only this sequential representation in Equation (3.4).

Table 1. First few polynomials $Q(x^n)$, for each degree n

n	$Q(x^n)$
1	$1\alpha_1$
2	$2\ \alpha_1\ x + 1\alpha_2$
3	$3\ \alpha_1\ x^2 + 3\ \alpha_2\ x + 1\alpha_3$
4	$4\ \alpha_1\ x^3 + 6\ \alpha_2\ x^2 + 4\ \alpha_3\ x + 1\alpha_4$
5	$5\ \alpha_1\ x^4 + 10\ \alpha_2\ x^3 + 10\ \alpha_3\ x^2 + 5\ \alpha_4\ x + 1\alpha_5$
6	$6\ \alpha_1\ x^5 + 15\ \alpha_2\ x^4 + 20\ \alpha_3\ x^3 + 15\ \alpha_4\ x^2 + 6\ \alpha_5\ x + 1\alpha_6$
7	$7\ \alpha_1\ x^6 + 21\ \alpha_2\ x^5 + 35\ \alpha_3\ x^4 + 35\ \alpha_4\ x^3 + 21\ \alpha_5\ x^2 + 7\ \alpha_6\ x^5 + 1\alpha_7$

Equation (3.4) in Theorem (4) is important in deriving many results for Sheffer as well as Appell set. The characterization of the delta operator is uniquely determined by the values of $\alpha_i's$ $(i = 1, 2, 3 \cdots n)$. In the next section, we study more about the delta operator in particular, the characterization of a delta operator which corresponds to a given Appell polynomials.

4 Appell Polynomials and Their Delta Operators

A new definition of Appell polynomials in [7], the generating function for two variable general Appel polynomials in [8] and the differential equations for Appel polynomials through the factorization metnod in [9] are effective study of characterization of the Appel polynomials. Several class of Appel-type polynomials which generalize the bernoulli and Euler are discussed in [10]. In this section, we obtain a theorem connecting the sequence of Appell polynomials for some delta operator. From the definitions (3) and (4), we see that an Appell sequence of polynomials is a Sheffer sequence for the delta operator D. Moreover, Every Appell sequence is a Sheffer sequence, but most Sheffer sequences are not Appell sequences. According to Rota [3], Appell polynomials are Sheffer polynomials relative to D.

Given a set of polynomials $p_n(x)$, with $p_0(x)$ is a non zero constant, under what conditions are they Appell polynomials ? A simple answer is given by

Theorem 5
If $p_n(x)$ is an Appell sequence for some delta operator Q, then the characterization of delta operator being : $\alpha_1 = 1$ *and* $\alpha_r = 0$ *for all* $r \geq 2$.

Proof:
Let $p_n(x)$ be a sequence of Appell polynomials.
By the sequential representation of delta operator Q in Equation (3.4), we see that

$$Q(x^n) = \sum_{r=1}^{n} \binom{n}{r}\ \alpha_r\ x^{n-r}.$$

The above equation can be written as

$$Q(x^n) = \sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ D^r\ x^n.$$

That is ,

$$Q \equiv \sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ D^r. \tag{4.1}$$

For simplicity, we write p_n instead of $p_n(x)$ and $p_n^{(r)}$ instead of $D^r p_n(x)$

$$Q(p_n) = \sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ p_n^{(r)}. \tag{4.2}$$

Since $p_n(x)$ is an Appell sequence, we have

$$Qp_n = Dp_n = np_{n-1} \tag{4.3}$$

Hence from the equations (4.2) and (4.3),

$$\sum_{r=1}^{n} \frac{1}{r!}\ \alpha_r\ p_n^{(r)} = np_{n-1}$$

That is,

$$\frac{1}{1!}\ \alpha_1\ D(p_n) + \frac{1}{2!}\ \alpha_2\ D^2(p_n) + \frac{1}{3!}\ \alpha_3\ D^3(p_n) + \cdots + \frac{1}{n!}\ \alpha_n\ D^n(p_n) = np_{n-1}$$

By equation (4.3),

$$\alpha_1\ np_{n-1} + \frac{1}{2!}\ \alpha_2\ D^2(p_n) + \frac{1}{3!}\ \alpha_3\ D^3(p_n) + \cdots + \frac{1}{n!}\ \alpha_n\ D^n(p_n) = np_{n-1}$$

By equating the corresponding terms, we get

$$\alpha_1 = 1\ \textit{and}\ \alpha_r = 0\ \textit{for all}\ r \geq 2$$

Thus we proved the theorem (5) □

The above theorem (5) gives a method to chart out the sequence of Appell polynomials and to find the corresponding delta operator. Here, by keeping the length of the present paper within bounds, we have choosen the following three Appell polynomials to verify the Theorem (5).

(i). The Bernoulli Polynomials

The generating function for the Bernoulli polynomials is

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x)\ \frac{t^n}{n!}$$

The n^{th} Bernoulli polynomial $B_n(x)$ is defined as follows :

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k\ x^{n-k},$$

for $n \geq 0$ and B_k, $(k = 0, 1, 2, \cdots)$ are the Bernoulli numbers.

Multiplying both sides by $e^t - 1$, we get

$$t[1 + tx + \frac{t^2x^2}{2!} + \cdots] = [t + \frac{t^2}{2!} + \cdots][B_0(x) + tB_1(x) + \frac{t^2}{2!}B_2(x) + \cdots]$$

Equating coefficients of various powers of t, we obtain

$$B_0(x) = 1$$
$$B_1(x) = x - \tfrac{1}{2}$$
$$B_2(x) = x^2 - x + \tfrac{1}{6}$$
$$B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x$$
$$B_4(x) = x^4 - 2x^3 + x^2 - \tfrac{1}{30}$$
$$B_5(x) = x^5 - \tfrac{5}{2}x^4 + \tfrac{5}{3}x^3 - \tfrac{1}{6}x \text{ and so on}$$

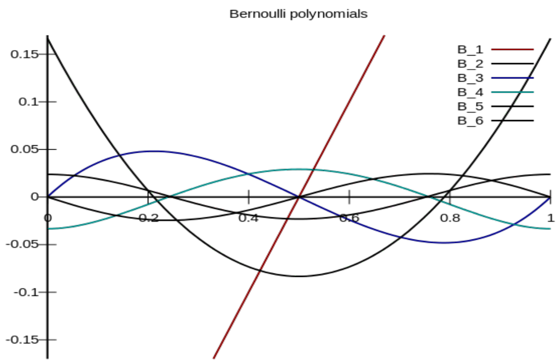

Fig 1. Bernoulli polynomials

For $n = 1$,

$QB_n = nB_{n-1}$ becomes $QB_1 = 1B_0$

From Table 1, we get $QB_1 = \alpha_1$ and $1B_0 = 1$

Thus we get

$$\alpha_1 = 1$$

For $n = 2$,

$QB_n = nB_{n-1}$ becomes $QB_2 = 2B_1$

By Table 1, we get $QB_2 = 2\alpha_1 x + \alpha_2 - \alpha_1$

and also $2B_1 = 2x - 1$

Equating the corresponding terms, we get

$$\alpha_1 = 1 \ \& \ \alpha_2 = 0$$

For $n = 3$,

$QB_n = nB_{n-1}$ becomes $QB_3 = 3B_2$

From Table 1, we get $QB_3 = 3\alpha_1 x^2 + 3\alpha_2 x - 3\alpha_1 x + \alpha_3 - \frac{3}{2}\alpha_2 + \frac{1}{2}\alpha_1$

And also $3B_2 = 3x^2 - 3x + \frac{1}{2}$

Equating the corresponding terms, we get

$$\alpha_1 = 1, \ \alpha_2 = 0 \ \& \ \alpha_3 = 0$$

Similarly proceeding as above ,

$$\alpha_1 = 1 \; and \; \alpha_2 = \alpha_3 = \cdots = \alpha_n = 0$$

Thus the characterization of the delta operator for Bernoulli polynomials being $\alpha_1 = 1$, and $\alpha_r = 0$ for all $r \geq 2$.
Thus the following Proposition holds.

Proposition 1.
For the Bernoulli polynomials $B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k \, x^{n-k}$, the characterization of the delta operator being $\alpha_1 = 1$, and $\alpha_r = 0$ for all $r \geq 2$. □

(ii). The Hermite Polynomials
The (probabilists) Hermite polynomials are given by

$$He_n(x) = (-1)^n \, e^{\frac{x^2}{2}} \, \frac{d^n}{dx^n} \, e^{\frac{-x^2}{2}}.$$

The Hermite polynomials constitute an Appell sequence, i.e., they are a polynomial sequence satisfying the following identity

$$\frac{d}{dx} He_n(x) = n \; He_{n-1}(x)$$

The first few Hermite polynomials are :

$$He_0(x) = 1$$
$$He_1(x) = x$$
$$He_2(x) = x^2 - 1$$
$$He_3(x) = x^3 - 3x$$
$$He_4(x) = x^4 - 6x^2 + 3$$
$$He_5(x) = x^5 - 10x^3 + 15x$$

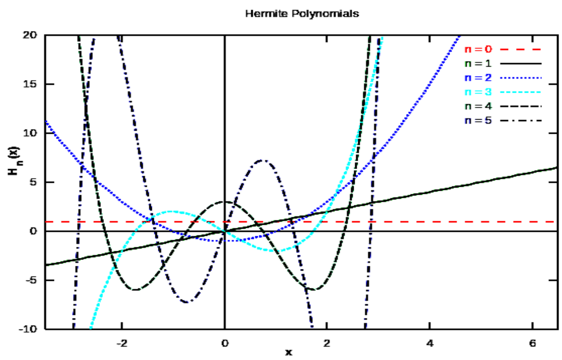

Fig 2. Hermite polynomials

For $n = 1$,

$$QHe_n = nHe_{n-1} \; becomes \; QHe_1 = 1He_0$$

From Table 1,

$$QHe_1 = \alpha_1 \ and \ 1He_0 = 1 \Rightarrow \alpha_1 = 1$$

For $n = 2$,

$$QHe_n = nHe_{n-1} \ becomes \ QHe_2 = 2He_1$$

By Table 1,

$$QHe_2 = 2\alpha_1 x + \alpha_2 \ and \ 2He_1 = 2x \Rightarrow \alpha_1 = 1 \ and \ \alpha_2 = 0$$

For $n = 3$,

$$QHe_n = nHe_{n-1} \ becomes \ QHe_3 = 3He_2$$

From Table 1,

$$QHe_3 = (3\alpha_1)x^2 + (3\alpha_2)x + \alpha_3 - 3\alpha_1 \ and \ 3He_2 = 3x^2 - 3$$

Equating the corresponding terms, we get

$$\alpha_1 = 1 \ , \ \alpha_2 = 0 \ and \ \alpha_3 = 0$$

By similar procedure, we get

$$\alpha_1 = 1 \ and \ \alpha_r = 0 \ for \ all \ r \geq 2$$

Hence the characterization of the delta operator for Hermite polynomials being $\alpha_1 = 1$, and $\alpha_r = 0$ for all $r \geq 2$.

Thus we obtain the following Proposition.

Proposition 1. For the Hermit's polynomial $He_n(x) = (-1)^n \ e^{\frac{x^2}{2}} \ \frac{d^n}{dx^n} \ e^{\frac{-x^2}{2}}$. the characterization of delta operator being $\alpha_1 = 1$, and $\alpha_r = 0$ for all $r \geq 2$. □

(iii). The Genocchi Polynomials

The Genocchi numbers are a sequence of integer that are defined by the exponential generating function :

$$\frac{2t}{e^t + 1} = e^{Gt} = \sum_{n=0}^{\infty} G_n \ \frac{t^n}{n!} \qquad (|t| < \phi)$$

When we multiply with e^{xt} in the left hand side , then we have

$$\sum_{n=0}^{\infty} G_n(x) \ \frac{t^n}{n!} = \frac{2t}{e^t + 1} \ e^{xt}, \qquad (|t| < \phi)$$

Here, $G_n(x)$ are called Genocchi polynomials. (see [11])

We now list a few Genocchi polynomials as follows :

$G_1(x) = 1$
$G_2(x) = 2x - 1$
$G_3(x) = 3x^2 - 3x$
$G_4(x) = 4x^3 - 6x^2 - 1$

Since $G_1(x)$ is a constant value 1, let us start from $n = 2$
For $n = 2$,

$$QG_n = nG_{n-1} \ becomes \ QG_2 = 2G_1$$

By Table 1,

$$QG_2 = 2\alpha_1 x \ and \ 2G_1 = 2 \Rightarrow \alpha_1 = 1$$

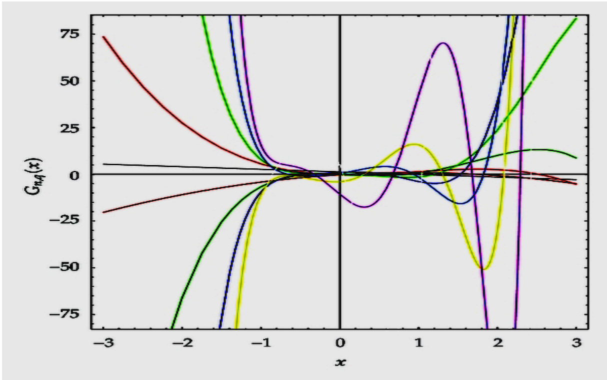

Fig 3. Genocchi polynomials

For $n = 3$,

$$QG_n = nG_{n-1} \text{ becomes } QG_3 = 3G_2$$

From Table 1,

$$QG_3 = 6\alpha_1 x + 3\alpha_2 - 3\alpha_1 \text{ and } 3G_2 = 6x - 3$$

Equating the corresponding terms, we get

$$\alpha_1 = 1 \text{ , and } \alpha_2 = 0$$

For $n = 4$,

$$QG_n = nG_{n-1} \text{ becomes } QG_4 = 4G_3$$

$$QG_4 = 12\alpha_1 x^2 + (12\alpha_2 - 12\alpha_1)x + 4\alpha_3 - 6\alpha_2 \text{ and } 4G_3 = 12x^2 - 12x$$

Equating the corresponding terms, we get

$$\alpha_1 = 1, \ \alpha_2 = 0 \text{ and } \alpha_3 = 0$$

By similar procedure, we get

$$\alpha_1 = 1 \text{ and } \alpha_r = 0 \text{ for all } r \geq 2$$

Hence the characterization of the delta operator for Genocchi polynomials being $\alpha_1 = 1$, and $\alpha_r = 0$ for all $r \geq 2$.

Thus we obtain the following Proposition.

Proposition 3. For the Genocchi polynomial $G_n(x)$, the characterization of delta operator being $\alpha_1 = 1$, and $\alpha_r = 0$ for all $r \geq 2$. □

From above three examples, we observe that the characterization of delta operator for Appell polynomials being $\alpha_1 = 1$, and $\alpha_r = 0$ for all $r \geq 2$ and hence the theorem (5) is verified.

5 Conclusions and Further Work

The properties of q-delta operator for q-basic polynomial sequence is discussed and analyzed in [12]. It is envisaged that the investigation of the characterization of q-delta operator for q- Appell polynomials may be developed in future.

Competing Interests

Authors have declared that no competing interests exist.

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