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Numerical Based Approach to Develop Analytical Solution of a Steady-state Isothermal Melt Spinning Model

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Authors' contributions

This work was carried out in collaboration between all authors. Author AIKB designed and supervised this study, did numerical computations. Authors WA and NA managed the calculations and analysis of the study, and performed literature searches. Author WA also wrote the first draft of this study. All authors read and approved the final manuscript.

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Method Article

Abstract

In this paper, we present a numerical based approach to develop an analytical solution of an isothermal melt spinning process modeled by a system of coupled non-linear ordinary differential equations. The obtained analytical solution is then compared with the numerical solution.

Keywords: Melt spinning process; isothermal; steady-state; shooting method.

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1 Introduction

At industrial scales, melt spinning is the most economical and convenient method for polymer fiber manufacturing. In the process of melt spinning, molten polymer is thrust out of the spinneret to form thin, cylindrical fibers. The drum placed at a distance from the spinneret is used to collect the fibers at a set take-up speed v_L. The take-up speed is considerably higher than extruding speed v_0 so that the fiber filament considerably becomes stretched in length and decreases its diameter between the take up point and the spinneret (die). The process is shown in the Fig. 1. The fiber is cooled between spinneret and take-up point such that it becomes solid at the take up point.

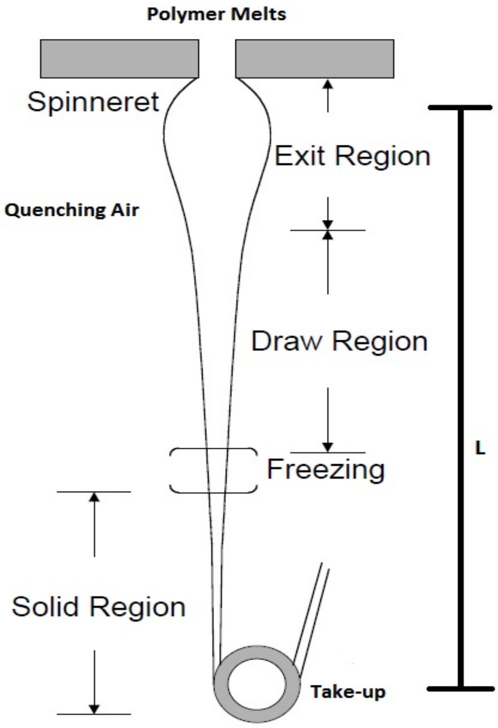

Fig. 1. Melt Spinning Process [2]

There are different mathematical models with different level of demands available in the literature that describe the process of melt spinning, e.g. see [1]-[6], [15]-[17]. In these research articles, the melt-spinning process has been considered for draw resonance and stability analysis. In [1], [2], [6] numerical solution of mathematical model describing the melt spinning process has been determined and used for stability analysis of the process and for optimizing the melt spinning processes. However, analytical solution of the said model has not been determined in any of these research articles.

In recent years, many mathematical models describing important physical and real world problems have been considered for obtaining some new properties such as solutions, physical meanings etc.(for example see [4], [7]-[14]). Thus, the concept of finding accurate numerical as well as analyitcal solutions of real world problems has attracted attention from all over the world.

In this article, we consider a steady-state model of an isothermal melt spinning process that consists of coupled non-linear ordinary differential equations of first and second order [2], [6] and present an approach to find an analytical solution of the model.

With the specified initial and boundary conditions associated with the mathematical models given in [2], [6], [16], it is not possible to determine an analytical solution of the model. The difficulty lies in finding the constants of integrations that appears in the general solution. To overcome this difficulty, we use a numerical approach called shooting method to convert the boundary value problem into a set of initial value problems (Detailed procedure of conversion is given in subsection 2.2). Once the model has been converted into a system of initial value problems, we use an existing integrating technique to determine the analytical solution.

The structure of the paper is as follows: In section 2, we give a brief description of the mathematical model of the isothermal melt spinning process. In section 3, the approach is presented and applied to find analytical solution of melt spinning process. The analytical and numerical solutions are compared through their graphs in section 4. Section 5 is devoted for conclusion.

2 Isothermal Melt Spinning Model

In the process of melt spinning, molten polymer is thrust out of the spinneret to form thin, cylindrical fibers. The drum placed at a distance from the spinneret is used to collect the fibers at a set take up speed. The take up speed is considerably higher than the extruding speed. The process is shown in the Fig. 1.

There are different mathematical models available in the literature that describe the process of melt spinning, e.g. see [2] and [6]. Here we consider a mathematical model for an isothermal melt spinning process [2] given as

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Av) = 0, \tag{2.1a}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \rho^{-1}\frac{1}{A}\frac{\partial}{\partial x}(A\omega) = 0, \tag{2.1b}$$

along with the initial and boundary conditions:

$$A(x=0,t) = A_0,\ v(x=0,t) = v_0,\ v(x=L,t) = v_L,\ \text{for all } t \geq 0, \tag{2.1c}$$

where L is the length of the spinline, t represents time and x denotes the coordinate along the spinline. System (2.1) gives us the velocity v and the cross-sectional area A of the fiber. The polymer density ρ is considered to be constant for isothermal process.

The constitutive model (2.1) needs a relation [2]

$$\omega = 3\eta\frac{dv}{dx}, \tag{2.2}$$

where ω is the axial stress and η is viscosity of melt polymer.

2.1 Dimensionless form

The dimensionless form of model (2.1) is obtained by introducing the following dimensionless scales

$$\tilde{t} = \frac{tv_0}{L},\ \tilde{v} = \frac{v}{v_0},\ \tilde{x} = \frac{x}{L},\ \tilde{A} = \frac{A}{A_0},\ \tilde{\omega} = \frac{\omega L}{\eta v_0},$$

into the model equations. The resulting dimensionless model equations are then given as

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Av) = 0, \tag{2.3a}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - \frac{3}{Re}\frac{1}{A}\frac{\partial}{\partial x}\left(A\frac{\partial v}{\partial x}\right) = 0, \tag{2.3b}$$

with the conditions

$$A(x=0,t) = 1,\ v(x=0,t) = 1,\ v(x=1,t) = d,\ \text{for all } t \geq 0, \tag{2.3c}$$

where we have removed the tilde notation for sake of simplicity, $Re = \frac{\rho L v_0}{\eta}$ is the dimensionless parameter known as Reynolds number and $d = \frac{v_L}{v_0} > 1$ denotes the draw ratio, an operating parameter of the melt spinning process. For the isothermal Newtonian flow of the melt spinning, the viscosity η of the melt polymer remains constant throughout the process and is considered as 1.

The steady state form of the model (2.3) is represented by a system of coupled nonlinear ordinary differential equations

$$\frac{d}{dx}(Av) = 0, \tag{2.4a}$$

$$-v\frac{dv}{dx} + \frac{3}{Re}\frac{1}{A}\frac{d}{dx}\left(A\frac{dv}{dx}\right) = 0, \tag{2.4b}$$

along with the conditions

$$A(x=0) = 1,\ v(x=0) = 1,\ v(x=1) = d. \tag{2.4c}$$

The system (2.4) represent a steady state form of an iso-thermal melt spinning process. We are interested to develop an analytical solution of this system.

2.2 Shooting method

We can solve boundary value problems analytically by different methods available in literature. The solution given by these methods contains constants of integration that can be easily found by using boundary conditions. However, there exist some problems (e.g problem (2.4)) where we are unable to find constants of integration with the given boundary conditions (see [2], [16]). In such a case, we may take help of a numerical method that transforms the boundary value problem into an initial value problem. *Shooting method* is the numerical technique, used to transform a boundary value problem of the type

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \quad y(a) = \alpha,\ y(b) = \beta,$$

into an initial value problem

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \quad y(a) = \alpha,\ \frac{dy}{dx}(a) = w, \tag{2.5}$$

where the number w is simply a guess and sometimes it is called an integrator. We apply one of the step-by-step numerical techniques like Euler's method or the MATLAB ODE solver *ode*45 (based on Runge-Kutta method) to the second order differential equation (2.5) to find an approximation w_i for the value of $y(b)$. Let the first guess w_1 be assumed as

$$w_1 = \frac{dy}{dz}(a) \approx \frac{y(b) - y(a)}{b - a}.$$

Let us consider N segments between the two boundaries $z = a$ and $z = b$ and define the step size $h = \dfrac{b-a}{N}$. We use Euler's method with w_1 to find an approximation β_1 for $y(b)$. If β_1 agrees with the given value $y(b)$ to some pre-assigned tolerance, we stop, otherwise consider another guess w_2 defined as

$$w_2 = 2w_1,$$

to obtain a second approximation β_2 for $y(b)$. If β_2 agrees with the value $y(b)$ to pre-assigned tolerance, we stop. If it does not agree, we use the secant approach to make further guesses using the formula

$$w_{i+1} = \frac{dy}{dz}(a) \approx w_i - e(w_i)\left(\frac{w_{i-1} - w_i}{e(w_{i-1}) - e(w_i)}\right), \quad i = 2, 3, \ldots \tag{2.6}$$

where

$$e(w_j) = \beta_j - y(b), \quad j = 1, 2, \ldots$$

be the error in the result of the shooting method.

Repeating the Euler's method with w_3 to obtain an approximation β_3 for $y(b)$. If β_3 agrees with the given value, we stop; otherwise, make a new guess w_4 using (2.6) to obtain an approximation β_4 for $y(b)$. This method can be continued in a trial and error manner until β_i agrees with the value $y(b)$ to pre-assigned tolerance.

Once we able to find a most suitable guess w, we can find an analytical solution of initial value problem of type given in (2.5).

3 Analytical Solution

To find an analytical solution of the steady state model (2.4), we proceed as follows:

By using (2.4c), solution of the continuity equation (2.4a) in terms of state variable v is given as

$$A = \frac{1}{v}, \ v > 0, \tag{3.1}$$

where A and v are functions of $x \in \Omega$. This solution shows a constant flow rate of the melt polymer.

Now, the equation (2.4b) can be put into the form

$$\frac{d^2v}{dx^2} = \frac{Re}{3}v\frac{dv}{dx} + \frac{1}{v}(\frac{dv}{dx})^2. \tag{3.2a}$$

This is a non-linear ordinary differential equation along with the conditions

$$v(x = 0) = 1, \ v(x = 1) = v_d. \tag{3.2b}$$

We transform the boundary value problem (3.2) into two initial value problems by setting

$$\frac{dv}{dx} = \frac{1}{3}\omega, \tag{3.3}$$

and hence

$$\frac{d^2v}{dx^2} = \frac{1}{9}\omega\frac{d\omega}{dv}.$$

The initial value problem corresponding to the boundary value problem (3.2) is then defined as

$$\frac{dv}{dx} = \frac{1}{3}\omega, \tag{3.4a}$$

$$\frac{d\omega}{dv} - \frac{1}{v}\omega = Rev, \tag{3.4b}$$

with the conditions

$$v(x=0) = 1,\ \omega(x=0) = \omega_0, \tag{3.4c}$$

where ω_0 is an approximated value guessed by shooting the value of v at 1 i.e. $v(1) = v_d$. (The procedure to obtain this approximation by shooting method has been explained in section 2).

Equation (3.4b) is integrated to get

$$\omega = Re\ v^2 + C_1 v, \tag{3.5}$$

where C_1 is a constant of integration, still to be determined.

Using (3.3), the solution (3.5) is put in the form

$$3\frac{dv}{dx} = Re\ v^2 + C_1 v. \tag{3.6}$$

Separating variables and then integrating, we have

$$\int \frac{1}{\frac{Re}{3}v^2 + \frac{1}{3}C_1 v} dv = x + C_2,$$

where C_2 is another constant of integration.

Solving the left hand integral, we reach at

$$\frac{3}{C_1}\ln\left|\frac{v}{v + \frac{1}{Re}C_1}\right| = x + C_2. \tag{3.7}$$

From equations (2.2), (2.4c) and (3.6), we get value of C_1 in the from

$$C_1 = \omega(0) - Re, \tag{3.8}$$

where $\omega(0)$ is the numerical value guessed by shooting the boundary $v(x=1) = d$ and $Re = \frac{\rho L v_0}{\eta}$ is the Reynold's number whose value is obtained using values given in Table 1.

In view of (2.4c), the equation (3.7) gives us the numerical value of C_2, i.e.

$$C_2 = -\frac{3}{C_1}\ln\left|1 + \frac{1}{Re}C_1\right|. \tag{3.9}$$

Equation (3.7) yields us

$$v = \frac{C_1 exp\left[\frac{1}{3}\left(C_1 x + C_1 C_2\right)\right]}{Re\left[1 - exp\left(\frac{1}{3}(C_1 x + C_1 C_2)\right)\right]}, \tag{3.10}$$

where v is the analytical form of steady state solution of the melt spinning model (2.4) along with C_1 and C_2 given in (3.8)-(3.9). Thus, the solutions A, obtained using (3.1), and v together constitute the analytical solution of the steady state model (2.4). Both of the solutions are plotted in the Fig. 2.

4 Comparison with Numerical Solution

To obtain the numerical solution of the steady-state model (2.4), we follow the strategy explained in section 2. The values of the parameters appearing in the steady state model equations (2.4) are given in the Table 1. Fig. 2 shows the numerical as well as the analytical solutions of the steady-state model (2.4) for velocity v and cross-sectional area A of the fiber along the spinline. We can observe that both of the solutions are overlapping each-other. The errors between the numerical and the analytical solutions for velocity v and cross-sectional area A, as shown in Fig. 3, are multiple of 10^{-7} and 10^{-5} respectively, which are negligibly small.

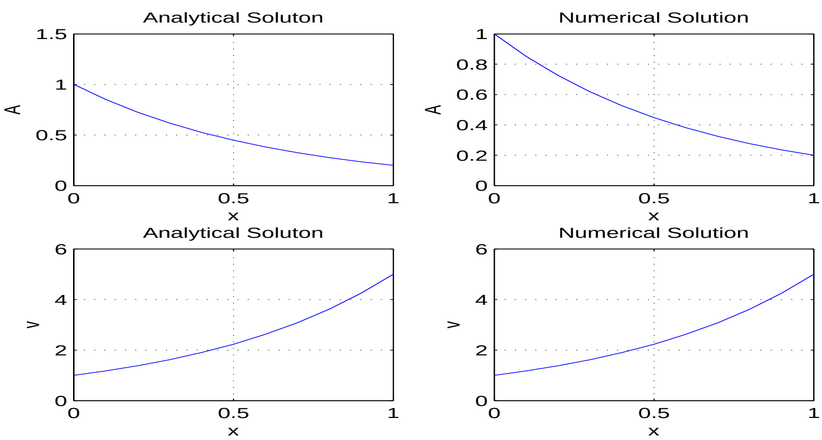

Fig. 2. Steady-state analytical and numerical solutions for velocity v and cross-sectional area A with $h = 0.01$

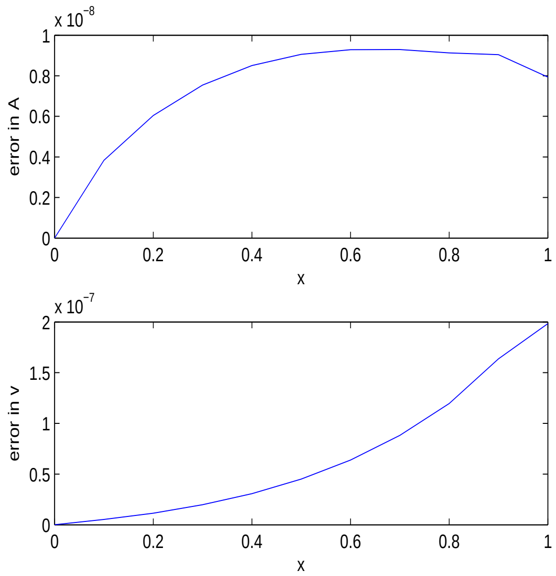

Fig. 3. Errors between analytical and numerical Steady-state solutions for velocity v and cross-sectional area A with $h = 0.01$

Table 1. Summary of parameteric values appearing in the steady state model (2.4)

Parameter	Symbol	Approximate Value	Units
Density of the polymer	ρ	0.001	kg/m^3
Feeding speed	v_0	10	m/s
Take up speed	v_L	50	m/s
Length of the spinline	L	1	m
Viscosity of the polymer	η	1	kg/ms

5 Conclusions

In this work we have used a numerical approach to find analytical solution of a boundary value problem where one cannot find constants of integrations with the given boundary conditions. The approach is to convert a boundary value problem into initial value problems and then to develop the analytical solution of the resulting problems with the usual methods. To explain the approach, we have developed an analytical solution of a steady-state model of an isothermal melt spinning process with the help of *shooting method.* The obtained analytical solutions are also plotted and compared with the numerical solution for error analysis.

Competing Interests

Authors have declared that no competing interests exist.

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