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Research Article

Levi-Civita Ricci-Flat Doubly Warped Product Hermitian Manifolds

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Let (M_1, g) and (M_2, h) be two Hermitian manifolds. The doubly warped product (abbreviated as DWP) Hermitian manifold of (M_1, g) and (M_2, h) is the product manifold $M_1 \times M_2$ endowed with the warped product Hermitian metric $G = f_2^2 g + f_1^2 h$, where f_1 and f_2 are positive smooth functions on M_1 and M_2, respectively. In this paper, the formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of the DWP-Hermitian manifold are derived in terms of the corresponding objects of its components. We also prove that if the warped function f_1 and f_2 are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat manifolds. Thus, we give an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

1. Introduction

It is well-known that the classification of various Ricci-flat manifolds are important topics in differential geometry. In 1967, Tani [1] first proposed the concept of Ricci-flat space in Riemannian geometry. Alvarez-Gaume and Freedman [2] showed that Ricci-flat space is a kind of space with great significance in theoretical physics, which attracted many scholars' research [3, 4]. In 1988, Bando and Kobayashi [5] characterized the Ricci-flat metric on Einstein-K*ä*hler manifold. In 2014, Liu and Yang [6] gave a sufficient and necessary condition for Hopf manifolds to be Levi-Civita Ricci-flat.

Levi-Civita connection is one of the most natural and effective tools for studying Riemannian manifolds [7]. In the complex case, Hsiung et al. [8] studied the general sectional curvature, the holomorphic sectional curvature, and holomorphic bisectional curvature of almost Hermitian manifolds by Levi-Civita connection and showed the relevance of above sectional curvatures. In 2012, Liu and Yang [8] gave Ricci-type curvatures and scalar curvatures of Hermitian manifolds by Levi-Civita connection (resp. Chern connection and Bismut connection) and obtained the relevance of these curvatures.

Warped product and twisted product are important methods used to construct manifold with special curvature properties in Riemann geometry and Finsler geometry. In Riemann geometry, Bishop and O'Neill [9] constructed Riemannian manifolds with negative curvature by warped product. Then, Brozos-Va'zquez et al. [10] used the warped product metrics to construct new examples of complete locally conformally flat manifolds with nonpositive curvature. After that, Leandro et al. [11] proved that an Einstein warped product manifold is a compact Riemannian manifold and its fibre is a Ricci-flat semi-Riemannian manifold.

On the other hand, warped product was extended to real Finsler geometry by the work of Asanov [12, 13]. In 2016, He and Zhong [14] generalized the warped product to complex Finsler geometry and proved that if complex Finsler manifold (M_1, F_1) and (M_2, F_2) are projectively flat, then the DWP-complex Finsler manifold is projectively flat if and only if the warped functions are positive constants. Moreover, He and Zhang [15] extended the doubly warped product to Hermitian case and got the Chern curvature, Chern Ricci curvature, and Chern Ricci scalar curvature of DWP-Hermitian manifold. They also gave the necessary and sufficient condition for a compact nontrivial DWP-Hermitian manifold to be of constant holomorphic sectional curvature. Recently, Xiao et al. [16] systematically studied holomorphic curvatures of doubly twisted product complex Finsler manifolds, and they [17] gave the necessary and sufficient condition for doubly twisted product complex Finsler manifold to be locally dually flat.

Thus, it is natural and interesting to ask the following question. Let (M_1, g) and (M_2, h) be two Levi-Civita Ricci-flat Hermitian manifolds, whether the DWP-Hermitian manifold is also a Levi-Civita Ricci-flat Hermitian manifold. Our purpose of doing this is to study the possibility of constructing Levi-Civita Ricci-flat manifold.

The structure of this paper is as follows. In Section 2, we briefly recall some basic concepts and notations which we need in this paper. In Section 3, we derive formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of DWP-Hermitian manifolds. In Section 4, we show that if the warped function f_1 and f_2 are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat manifolds.

2. Preliminary

Let (M, J, G) be a Hermitian manifold with $\dim_{\mathbb{C}} M = n$; here, J is the complex structure, and G is a Hermitian metric. For a point $p \in M$, the complexified tangent bundle $T_p^{\mathbb{C}} M = T_p M \otimes C$ is decomposed as

$$T_p^{\mathbb{C}} M = T_p^{1,0} M \oplus T_p^{0,1} M, \quad (1)$$

where $T_p^{1,0} M$ and $T_p^{0,1} M$ are the eigenspaces of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

In this paper, we set $\partial_\alpha = \partial/\partial z^\alpha$ and $\partial_{\bar{\alpha}} = \partial/\partial \bar{z}^{\alpha}$. Let $z = (z^1, \cdots, z^n)$ be the local holomorphic coordinates on M; then, the vector fields $(\partial_1, \cdots, \partial_n)$ form a basis for $T_p^{1,0} M$. Levi-Civita connection ∇^{LC} on the holomorphic tangent bundle $T_p^{1,0} M$ is defined by [18]

$$\nabla^{LC} = \pi \circ \nabla : \Gamma\left(M, T^{1,0} M\right) \xrightarrow{\nabla} \Gamma\left(M, T_p M \otimes T_p M\right) \xrightarrow{\pi} \Gamma\left(M, T_p M \otimes T^{1,0} M\right). \quad (2)$$

In local coordinate system, its connection is as follows [18]:

$$\begin{aligned} \nabla^{LC}_{\partial/\partial z^\alpha} \frac{\partial}{\partial z^\beta} £^\circ &= \Gamma^\gamma_{\alpha\beta} \frac{\partial}{\partial z^\gamma}, \\ \nabla^{LC}_{\partial/\partial \bar{z}^\varepsilon} \frac{\partial}{\partial z^\beta} £^\circ &= \Gamma^\gamma_{\bar{\varepsilon}\beta} \frac{\partial}{\partial z^\gamma}, \end{aligned} \quad (3)$$

where the Levi-Civita connection coefficients $\Gamma^\gamma_{\alpha\beta}$ and $\Gamma^\gamma_{\bar{\alpha}\beta}$ are given by [18]

$$\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} G^{\gamma\bar{\varepsilon}} \left(\partial_\alpha G_{\beta\bar{\varepsilon}} + \partial_\beta G_{\alpha\bar{\varepsilon}}\right), \quad (4)$$

$$\Gamma^\gamma_{\bar{\alpha}\beta} = \frac{1}{2} G^{\gamma\bar{\varepsilon}} \left(\partial_{\bar{\alpha}} G_{\beta\bar{\varepsilon}} - \partial_{\bar{\varepsilon}} G_{\beta\bar{\alpha}}\right). \quad (5)$$

Let $K \in \Gamma(M, \Lambda^2 T_p M \otimes T^{*1,0} M \otimes T^{1,0} M)$ be the Levi-Civita curvature tensor such as

$$K(X, Y)s = \nabla^{LC}_X \nabla^{LC}_Y s - \nabla^{LC}_Y \nabla^{LC}_X s - \nabla^{LC}_{[X,Y]} s, \quad (6)$$

where $X, Y \in T_p M$, $s \in T^{1,0} M$. In the local coordinate system, the coefficients of K are given by

$$K^\varepsilon_{\alpha\bar{\beta}\gamma} = -\left[\partial_{\bar{\beta}} \Gamma^\varepsilon_{\alpha\gamma} - \partial_\alpha \Gamma^\varepsilon_{\bar{\beta}\gamma} + \Gamma^\lambda_{\alpha\gamma} \Gamma^\varepsilon_{\bar{\beta}\lambda} - \Gamma^\lambda_{\bar{\beta}\gamma} \Gamma^\varepsilon_{\lambda\alpha}\right]. \quad (7)$$

Definition 1 (see [6]). The first Levi-Civita Ricci curvature $K^{(1)}$ on the Hermitian manifold (M, J, G) is defined by

$$K^{(1)} = \sqrt{-1} K^{(1)}_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta, \quad (8)$$

where

$$K^{(1)}_{\alpha\bar{\beta}} = G^{\gamma\bar{\delta}} K_{\alpha\bar{\beta}\gamma\bar{\delta}}, \quad (9)$$

$$K_{\alpha\bar{\beta}\gamma\bar{\delta}} = G_{\varepsilon\bar{\delta}} K^\varepsilon_{\alpha\bar{\beta}\gamma}. \quad (10)$$

Levi-Civita Ricci scalar curvature S_{LC} on $T^{1,0} M$ is given by

$$S_{LC} = G^{\alpha\bar{\beta}} K^{(1)}_{\alpha\bar{\beta}}. \quad (11)$$

Definition 2 (see [6]). Hermitian metric G on M is called Levi-Civita Ricci-flat if

$$K^{(1)}(G) = 0. \quad (12)$$

Let (M_1, g) and (M_2, h) be two Hermitian manifolds with $\dim_{\mathbb{C}} M_1 = m$ and $\dim_{\mathbb{C}} M_2 = n$; then, $M = M_1 \times M_2$ is a Hermitian manifold with $\dim_{\mathbb{C}} M = m + n$.

Denote $\pi_1 : M \longrightarrow M_1$ and $\pi_2 : M \longrightarrow M_2$ the natural projections. Note that $\pi_1(z) = z_1$ and $\pi_2(z) = z_2$ for every $z = (z_1, z_2) \in M$ with $z_1 = (z^1, \cdots, z^m) \in M_1$ and $z_2 = (z^{m+1}, \cdots, z^{m+n}) \in M_2$.

Denote $d\pi_1 : T^{1,0}(M) \longrightarrow T^{1,0} M_1$, $d\pi_2 : T^{1,0}(M) \longrightarrow T^{1,0} M_2$ the holomorphic tangent maps induced by π_1 and π_2, respectively. Note that $d\pi_1(z, v) = (z_1, v_1)$ and $d\pi_2(z, v) = (z_2, v_2)$ for every $v = (v_1, v_2) \in T_z^{1,0}(M)$ with $v_1 = (v^1, \cdots, v^m) \in T^{1,0}_{z_1} M_1$ and $v_2 = (v^{m+1}, \cdots, v^{m+n}) \in T^{1,0}_{z_2} M_2$.

Definition 3 (see [15]). Let (M_1, g) and (M_2, h) be two Hermitian manifolds. $f_1 : M_1 \longrightarrow (0, +\infty)$ and $f_2 : M_2 \longrightarrow (0, +\infty)$ be two positive smooth functions. The doubly warped product (abbreviated as DWP) Hermitian manifold $({}_{f_2}M_1 \times_{f_1} M_2, G)$ is the product Hermitian manifold $M = M_1 \times M_2$ endowed with the Hermitian metric $G : M \longrightarrow \mathbb{R}^+$

defined by

$$G(z,v) = (f_2 \circ \pi_2)^2(z) g(\pi_1(z), d\pi_1(v)) + (f_1 \circ \pi_1)^2(z) h(\pi_2(z), d\pi_2(v)), \tag{13}$$

for $z = (z_1, z_2) \in M$ and $v = (v_1, v_2) \in T_z^{1,0} M$. f_1 and f_2 are warped functions; the DWP-Hermitian manifold of (M_1, g) and (M_2, h) is denoted by $({}_{f_2}M_1 \times_{f_1} M_2, G)$.

If either $f_1 = 1$ or $f_2 = 1$, then $({}_{f_2}M_1 \times_{f_1} M_2, G)$ becomes a warped product of Hermitian manifolds (M_1, g) and (M_2, h). If $f_1 \equiv 1$ and $f_2 \equiv 1$, then $({}_{f_2}M_1 \times_{f_1} M_2, G)$ becomes a product of Hermitian manifolds (M_1, g) and (M_2, h). If neither f_1 nor f_2 is constant, then we call $({}_{f_2}M_1 \times_{f_1} M_2, G)$ a nontrivial DWP-Hermitian manifolds of (M_1, g) and (M_2, h).

Notation 4. Lowercase Greek indices such as α, β, and γ will run from 1 to $m+n$, lowercase Latin indices such as i, j, and k will run from 1 to m, and lowercase Latin indices with a prime, such as i', j', and k', will run from $m+1$ to $m+n$. Quantities associated to (M_1, g) and (M_2, h) are denoted with upper indices 1 and 2, respectively, such as $\overset{1}{\Gamma^i_{jk}}$ and $\overset{2}{\Gamma^{i'}_{j'k'}}$ are Levi-Civita connection coefficients of (M_1, g) and (M_2, h), respectively.

Denote

$$\begin{aligned} g_{i\bar{j}} &= \frac{\partial^2 g}{\partial v^i \partial \overline{v^j}}, \\ h_{i'\bar{j}'} &= \frac{\partial^2 h}{\partial v^{i'} \partial \overline{v^{j'}}}. \end{aligned} \tag{14}$$

The fundamental tensor matrix of G is given by

$$\left(G_{\alpha\bar{\beta}}\right) = \left(\frac{\partial^2 G}{\partial v^\alpha \partial \overline{v^\beta}}\right) = \begin{pmatrix} f_2^2 g_{i\bar{j}} & 0 \\ 0 & f_1^2 h_{i'\bar{j}'} \end{pmatrix}, \tag{15}$$

and its inverse matrix $(G^{\bar{\beta}\alpha})$ is given by

$$\left(G^{\bar{\beta}\alpha}\right) = \begin{pmatrix} f_2^{-2} g^{\bar{j}i} & 0 \\ 0 & f_1^{-2} h^{\bar{j}'i'} \end{pmatrix}. \tag{16}$$

Proposition 5. *Let $({}_{f_2}M_1 \times_{f_1} M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then, the Levi-Civita connection coefficients $\Gamma^\gamma_{\alpha\beta}$ associated to G are given by*

$$\begin{aligned} \Gamma^k_{ij} &= \overset{1}{\Gamma^k_{ij}}, \\ \Gamma^k_{i'j} &= f_2^{-1} \frac{\partial f_2}{\partial z^{i'}} \delta^k_j, \\ \Gamma^k_{ij'} &= f_2^{-1} \frac{\partial f_2}{\partial z^{j'}} \delta^k_i, \\ \Gamma^{k'}_{i'j'} &= \overset{2}{\Gamma^{k'}_{i'j'}}, \\ \Gamma^{k'}_{ij'} &= f_1^{-1} \frac{\partial f_1}{\partial z^i} \delta^{k'}_{j'}, \\ \Gamma^{k'}_{i'j} &= f_1^{-1} \frac{\partial f_1}{\partial z^j} \delta^{k'}_{i'}, \\ \Gamma^k_{i'j'} &= \Gamma^{k'}_{ij} = 0. \end{aligned} \tag{17}$$

Proof. Substituting (15) and (16) into (4), we obtain

$$\begin{aligned} \Gamma^k_{ij} &= \frac{1}{2} G^{k\bar{\varepsilon}} \left(\partial_i G_{j\bar{\varepsilon}} + \partial_j G_{i\bar{\varepsilon}}\right) + \frac{1}{2} G^{k\overline{\varepsilon'}} \left(\partial_i G_{j\overline{\varepsilon'}} + \partial_j G_{i\overline{\varepsilon'}}\right) \\ &= \frac{1}{2} G^{k\bar{l}} \left(\frac{\partial G_{j\bar{l}}}{\partial z^i} + \frac{\partial G_{i\bar{l}}}{\partial z^j}\right) + \frac{1}{2} G^{k\overline{l'}} \left(\frac{\partial G_{j\overline{l'}}}{\partial z^i} + \frac{\partial G_{i\overline{l'}}}{\partial z^j}\right) \\ &= \frac{1}{2} f_2^{-2} g^{k\bar{l}} \left(2 f_2 \frac{\partial f_2}{\partial z^i} g_{j\bar{l}} + f_2^2 \frac{\partial g_{j\bar{l}}}{\partial z^i} + 2 f_2 \frac{\partial f_2}{\partial z^j} g_{i\bar{l}} + f_2^2 \frac{\partial g_{i\bar{l}}}{\partial z^j}\right) \\ &= \frac{1}{2} g^{k\bar{l}} \left(\frac{\partial g_{j\bar{l}}}{\partial z^i} + \frac{\partial g_{i\bar{l}}}{\partial z^j}\right) = \overset{1}{\Gamma^k_{ij}}. \end{aligned} \tag{18}$$

Similarly, we can obtain other equations of Proposition 5. □

Plugging (15) and (16) into (5), we have the following proposition.

Proposition 6. *Let $({}_{f_2}M_1 \times_{f_1} M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then, the Levi-Civita connection coefficients $\Gamma^\gamma_{\bar{\alpha}\beta}$ associated to G are given by*

$$\begin{aligned} \Gamma^k_{\bar{i}j} &= \overset{1}{\Gamma^k_{\bar{i}j}}, \\ \Gamma^k_{\bar{i}'j'} &= -f_2^{-2} f_1 g^{k\bar{l}} \frac{\partial f_1}{\partial \bar{z}^l} h_{j'\overline{i'}}, \\ \Gamma^k_{\bar{i}'j} &= f_2^{-1} \frac{\partial f_2}{\partial \bar{z}^{i'}} \delta^k_j, \\ \Gamma^{k'}_{\bar{i}'j'} &= \overset{2}{\Gamma^{k'}_{\bar{i}'j'}}, \\ \Gamma^{k'}_{\bar{i}j} &= -f_1^{-2} f_2 h^{k'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} g_{j\bar{i}}, \\ \Gamma^{k'}_{\bar{i}j'} &= f_1^{-1} \frac{\partial f_1}{\partial \bar{z}^i} \delta^{k'}_{j'}, \\ \Gamma^k_{\bar{i}j'} &= \Gamma^{k'}_{\bar{i}'j} = 0. \end{aligned} \tag{19}$$

3. Levi-Civita Ricci Scalar Curvature of Doubly Warped Product Hermitian Manifolds

In this section, we derive formulae of Levi-Civita curvature, Levi-Civita Ricci curvature, and Levi-Civita Ricci scalar curvature of DWP-Hermitian manifold.

Proposition 7. *Let* $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ *be a DWP-Hermitian manifold of* (M_1, g) *and* (M_2, h)*. Then, the coefficients of Levi-Civita curvature tensor* $K^{\varepsilon}_{\alpha\bar{\beta}\gamma}$ *are given by*

$$K^t_{k\bar{j}s} = \overset{1}{K}{}^t_{k\bar{j}s} + f_1^{-2} h^{i'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_2}{\partial z^{i'}} g_{s\bar{j}} \delta^t_k, \tag{20}$$

$$K^{t'}_{k'\overline{j'}s'} = \overset{2}{K}{}^{t'}_{k'\overline{j'}s'} + f_2^{-2} g^{i\bar{l}} \frac{\partial f_1}{\partial \bar{z}^{l}} \frac{\partial f_1}{\partial z^{i}} h_{s'\overline{j'}} \delta^{t'}_{k'}, \tag{21}$$

$$K^t_{k'\overline{j'}s} = f_2^{-2} g^{t\bar{l}} \frac{\partial f_1}{\partial \bar{z}^{l}} \frac{\partial f_1}{\partial z^{s}} h_{k'\overline{j'}}, \tag{22}$$

$$K^{t'}_{k\bar{j}s'} = f_1^{-2} h^{t'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_2}{\partial z^{s'}} g_{k\bar{j}}, \tag{23}$$

$$K^t_{k\bar{j}s'} = f_1^{-1} f_2^{-1} \frac{\partial f_1}{\partial \bar{z}^{j}} \frac{\partial f_2}{\partial z^{i'}} \delta^{i'}_{s'} \delta^t_k - \frac{1}{2} f_2^{-1} \frac{\partial f_2}{\partial z^{s'}} g^{t\bar{l}} \left(\frac{\partial g_{i\bar{l}}}{\partial \bar{z}^{j}} - \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^{l}} \right) \delta^i_k, \tag{24}$$

$$K^{t'}_{k'\overline{j'}s} = f_1^{-1} f_2^{-1} \frac{\partial f_1}{\partial z^{i}} \frac{\partial f_2}{\partial \bar{z}^{j'}} \delta^i_s \delta^{t'}_{k'} - \frac{1}{2} f_1^{-1} \frac{\partial f_1}{\partial z^{s}} h^{i'\overline{l'}} \left(\frac{\partial h_{i'\overline{l'}}}{\partial \bar{z}^{j'}} - \frac{\partial h_{i'\overline{l'}}}{\partial \bar{z}^{l'}} \right) \delta^{i'}_{k'}, \tag{25}$$

$$K^t_{k'\overline{j'}s'} = f_2^{-3} f_1 g^{t\bar{l}} \frac{\partial f_1}{\partial \bar{z}^{l}} \frac{\partial f_2}{\partial z^{k'}} h_{s'\overline{j'}} + f_2^{-2} f_1 g^{t\bar{l}} \frac{\partial f_1}{\partial \bar{z}^{l}} \left[\frac{1}{2} \left(\frac{\partial h_{k'\overline{l'}}}{\partial z^{s'}} + \frac{\partial h_{s'\overline{l'}}}{\partial z^{k'}} \right) \delta^{\overline{l'}}_{\overline{j'}} + \frac{\partial h_{s'\overline{j'}}}{\partial z^{k'}} \right]. \tag{26}$$

$$K^{t'}_{k\bar{j}s} = f_1^{-3} f_2 h^{t'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_1}{\partial z^{k}} g_{s\bar{j}} + f_1^{-2} f_2 h^{t'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \left[\frac{1}{2} \left(\frac{\partial g_{k\bar{l}}}{\partial z^{s}} + \frac{\partial g_{s\bar{l}}}{\partial z^{k}} \right) \delta^{\bar{l}}_{\bar{j}} + \frac{\partial g_{s\bar{j}}}{\partial z^{k}} \right], \tag{27}$$

$$\begin{aligned} K^t_{k\overline{j'}s'} = {} & \frac{\partial^2 \ln f_2}{\partial z^{s'} \partial \bar{z}^{j'}} \delta^t_k + \frac{1}{2} f_2^{-1} h^{i'\overline{l'}} \frac{\partial f_2}{\partial z^{i'}} \left(\frac{\partial h_{s'\overline{l'}}}{\partial \bar{z}^{j'}} - \frac{\partial h_{s'\overline{j'}}}{\partial \bar{z}^{l'}} \right) \delta^t_k \\ & + f_2^{-2} f_1 h_{s'\overline{j'}} \left[\frac{\partial g^{t\bar{l}}}{\partial z^{k}} \frac{\partial f_1}{\partial \bar{z}^{l}} - g^{t\bar{l}} \frac{\partial^2 f_1}{\partial z^k \partial \bar{z}^l} - \frac{1}{2} g^{i\bar{l}} g^{t\bar{l}} \frac{\partial f_1}{\partial \bar{z}^{l}} \left(\frac{\partial g_{k\bar{l}}}{\partial z^{i}} + \frac{\partial g_{i\bar{l}}}{\partial z^{k}} \right) \right], \end{aligned} \tag{28}$$

$$\begin{aligned} K^{t'}_{k'\bar{j}s} = {} & \frac{\partial^2 \ln f_1}{\partial z^{s} \partial \bar{z}^{j}} \delta^{t'}_{k'} + \frac{1}{2} f_1^{-1} g^{i\bar{l}} \frac{\partial f_1}{\partial z^{i}} \left(\frac{\partial g_{s\bar{l}}}{\partial \bar{z}^{j}} - \frac{\partial g_{s\bar{j}}}{\partial \bar{z}^{l}} \right) \delta^{t'}_{k'} \\ & + f_1^{-2} f_2 g_{s\bar{j}} \left[\frac{\partial h^{t'\overline{l'}}}{\partial z^{k'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} - h^{t'\overline{l'}} \frac{\partial^2 f_2}{\partial z^{k'} \partial \bar{z}^{l'}} - \frac{1}{2} h^{i'\overline{l'}} h^{t'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \left(\frac{\partial h_{k'\overline{l'}}}{\partial z^{i'}} + \frac{\partial h_{i'\overline{l'}}}{\partial z^{k'}} \right) \right]. \end{aligned} \tag{29}$$

$$K^t_{k'\bar{j}s} = K^t_{k'\bar{j}s'} = K^t_{k\overline{j'}s} = K^{t'}_{k\overline{j'}s} = K^{t'}_{k'\bar{j}s'} = K^{t'}_{k\overline{j'}s'} = 0. \tag{30}$$

Proof. Using (7), we have

$$K^t_{k\bar{j}s} = -\left[\partial_{\bar{j}} \Gamma^t_{ks} - \partial_k \Gamma^t_{\bar{j}s} + \Gamma^{\lambda}_{ks} \Gamma^t_{\bar{j}\lambda} - \Gamma^{\lambda}_{\bar{j}s} \Gamma^t_{\lambda k} + \Gamma^{\lambda'}_{ks} \Gamma^t_{\bar{j}\lambda'} - \Gamma^{\lambda'}_{\bar{j}s} \Gamma^t_{\lambda' k} \right]. \tag{31}$$

Taking the formulae of Proposition 5 and Proposition 6 into (31), we obtain

$$\begin{aligned} K^t_{k\bar{j}s} & = -\left[\frac{\partial \overset{1}{\Gamma}{}^t_{ks}}{\partial \bar{z}^j} - \frac{\partial \overset{1}{\Gamma}{}^t_{\bar{j}s}}{\partial z^k} + \overset{1}{\Gamma}{}^i_{ks} \overset{1}{\Gamma}{}^t_{\bar{j}i} - \overset{1}{\Gamma}{}^i_{\bar{j}s} \overset{1}{\Gamma}{}^t_{ik} + \Gamma^{i'}_{ks} \Gamma^t_{\bar{j}i'} - \Gamma^{i'}_{\bar{j}s} \Gamma^t_{i'k} \right] \\ & = \overset{1}{K}{}^t_{k\bar{j}s} + f_1^{-2} h^{i'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_2}{\partial z^{i'}} g_{s\bar{j}} \delta^t_k. \end{aligned} \tag{32}$$

Similarly, we can obtain other equations of Proposition 7. □

Proposition 8. *Let* $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ *be a DWP-Hermitian manifold of* (M_1, g) *and* (M_2, h)*. Then,*

$$K_{k\bar{j}s\bar{p}} = f_2^2 \overset{1}{K}_{k\bar{j}s\bar{p}} + f_1^{-2} f_2^2 h^{i'\overline{l'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_2}{\partial z^{i'}} g_{s\bar{j}} g_{k\bar{p}},$$

$$K_{k'\overline{j'}s'\overline{p'}} = f_1^2 \overset{2}{K}_{k'\overline{j'}s'\overline{p'}} + f_2^{-2} f_1^2 g^{i\bar{l}} \frac{\partial f_1}{\partial \bar{z}^{l}} \frac{\partial f_1}{\partial z^{i}} h_{s'\overline{j'}} h_{k'\overline{p'}},$$

$$K_{k\bar{j}s'\overline{p'}} = \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_2}{\partial z^{s'}} g_{k\bar{j}} \delta^{\overline{l'}}_{\overline{p'}},$$

$$K_{k'\overline{j'}s\bar{p}} = \frac{\partial f_1}{\partial \bar{z}^{l}} \frac{\partial f_1}{\partial z^{s}} h_{k'\overline{j'}} \delta^{\bar{l}}_{\bar{p}},$$

$$K_{k\bar{j}s'\bar{p}} = f_1^{-1} f_2 g_{k\bar{p}} \frac{\partial f_1}{\partial \bar{z}^{j}} \frac{\partial f_2}{\partial z^{i'}} \delta^{i'}_{s'} - \frac{1}{2} f_2 \frac{\partial f_2}{\partial z^{s'}} \left(\frac{\partial g_{i\bar{l}}}{\partial \bar{z}^{j}} - \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^{l}} \right) \delta^i_k \delta^{\bar{l}}_{\bar{p}},$$

$$K_{k'\overline{j'}s\overline{p'}} = f_1 f_2^{-1} h_{k'\overline{p'}} \frac{\partial f_1}{\partial z^{i}} \frac{\partial f_2}{\partial \bar{z}^{j'}} \delta^i_s - \frac{1}{2} f_1 \frac{\partial f_1}{\partial z^{s}} \left(\frac{\partial h_{i'\overline{l'}}}{\partial \bar{z}^{j'}} - \frac{\partial h_{i'\overline{j'}}}{\partial \bar{z}^{l'}} \right) \delta^{i'}_{k'} \delta^{\overline{l'}}_{\overline{p'}},$$

$$K_{k\bar{j}s\overline{p'}} = f_1^{-3} f_2 \frac{\partial f_2}{\partial \bar{z}^{l'}} \frac{\partial f_1}{\partial z^{k}} g_{s\bar{j}} \delta^{\overline{l'}}_{\overline{p'}} + f_2 \frac{\partial f_2}{\partial \bar{z}^{l'}} \left[\frac{1}{2} \left(\frac{\partial g_{k\bar{l}}}{\partial z^{s}} + \frac{\partial g_{s\bar{l}}}{\partial z^{k}} \right) \delta^{\bar{l}}_{\bar{j}} + \frac{\partial g_{s\bar{j}}}{\partial z^{k}} \right] \delta^{\overline{l'}}_{\overline{p'}},$$

$$K_{k'\overline{j'}s'\bar{p}} = f_2^{-3} f_1 \frac{\partial f_1}{\partial \bar{z}^{l}} \frac{\partial f_2}{\partial z^{k'}} h_{s'\overline{j'}} \delta^{\bar{l}}_{\bar{p}} + f_1 \frac{\partial f_1}{\partial \bar{z}^{l}} \left[\frac{1}{2} \left(\frac{\partial h_{k'\overline{l'}}}{\partial z^{s'}} + \frac{\partial h_{s'\overline{l'}}}{\partial z^{k'}} \right) \delta^{\overline{l'}}_{\overline{j'}} + \frac{\partial h_{s'\overline{j'}}}{\partial z^{k'}} \right] \delta^{\bar{l}}_{\bar{p}},$$

$$\begin{aligned} K_{k'\bar{j}s\overline{p'}} = {} & f_1^2 h_{k'\overline{p'}} \frac{\partial^2 \ln f_1}{\partial z^s \partial \bar{z}^j} + \frac{1}{2} f_1 h_{k'\overline{p'}} g^{i\bar{l}} \frac{\partial f_1}{\partial z^{i}} \left(\frac{\partial g_{s\bar{l}}}{\partial \bar{z}^{j}} - \frac{\partial g_{s\bar{j}}}{\partial \bar{z}^{l}} \right) \\ & + f_2 g_{s\bar{j}} \left[\frac{\partial h^{t'\overline{l'}}}{\partial z^{k'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} h_{t'\overline{p'}} + \delta^{\overline{l'}}_{\overline{p'}} \frac{\partial^2 f_2}{\partial z^{k'} \partial \bar{z}^{l'}} \right. \\ & \left. + \frac{1}{2} h^{i'\overline{l'}} \delta^{\overline{l'}}_{\overline{p'}} \frac{\partial f_2}{\partial \bar{z}^{l'}} \left(\frac{\partial h_{k'\overline{l'}}}{\partial z^{i'}} + \frac{\partial h_{i'\overline{l'}}}{\partial z^{k'}} \right) \right], \end{aligned}$$

$$\begin{aligned} K_{k\overline{j'}s'\bar{p}} = {} & f_2^2 g_{k\bar{p}} \frac{\partial^2 \ln f_2}{\partial z^{s'} \partial \bar{z}^{j'}} + \frac{1}{2} f_2 g_{k\bar{p}} h^{i'\overline{l'}} \frac{\partial f_2}{\partial z^{i'}} \left(\frac{\partial h_{s'\overline{l'}}}{\partial \bar{z}^{j'}} - \frac{\partial h_{s'\overline{j'}}}{\partial \bar{z}^{l'}} \right) \\ & + f_1 h_{s'\overline{j'}} \left[\frac{\partial g^{t\bar{l}}}{\partial z^{k}} \frac{\partial f_1}{\partial \bar{z}^{l}} g_{t\bar{p}} + \delta^{\bar{l}}_{\bar{p}} \frac{\partial^2 f_1}{\partial z^k \partial \bar{z}^l} + \frac{1}{2} g^{i\bar{l}} \delta^{\bar{l}}_{\bar{p}} \frac{\partial f_1}{\partial \bar{z}^{l}} \left(\frac{\partial g_{k\bar{l}}}{\partial z^{i}} + \frac{\partial g_{i\bar{l}}}{\partial z^{k}} \right) \right]. \end{aligned}$$

$$K_{k'\bar{j}s'\bar{p}} = K_{k'\bar{j}s\bar{p}} = K_{k'\bar{j}s'\overline{p'}} = K_{k\overline{j'}s\bar{p}} = K_{k\overline{j'}s\overline{p'}} = K_{k\overline{j'}s'\overline{p'}} = 0. \tag{33}$$

Proof. According to (10), we get

$$K_{k\bar{j}s\bar{p}} = G_{\varepsilon\bar{p}}K^{\varepsilon}_{k\bar{j}s} = G_{t\bar{p}}K^{t}_{k\bar{j}s} + G_{t'\bar{p}}K^{t'}_{k\bar{j}s}. \tag{34}$$

Substituting (20), (27), and (15) into (34), we have

$$K_{k\bar{j}s\bar{p}} = G_{t\bar{p}}\left(\overset{1}{K^{t}_{k\bar{j}s}} + f_1^{-2}h^{i'\bar{l}'}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{s\bar{j}}\delta^t_k\right) = f_2^2\overset{1}{K_{k\bar{j}s\bar{p}}} + f_1^{-2}f_2^2h^{i'\bar{l}'}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{s\bar{j}}g_{k\bar{p}}. \tag{35}$$

Similarly, we can obtain other equations of Proposition 8. □

Proposition 9. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then, the coefficients of the first Levi-Civita Ricci curvature $K^{(1)}_{\alpha\bar{\beta}}$ are given by*

$$\begin{cases} K^{(1)}_{k\bar{j}} = \overset{1}{K^{(1)}_{k\bar{j}}} + 2f_1^{-2}\dfrac{\partial f_2}{\partial \bar{z}^{l'}}\dfrac{\partial f_2}{\partial z^{i'}}g_{k\bar{j}}h^{i'\bar{l}'}, \\ K^{(1)}_{k'\bar{j}'} = \overset{2}{K^{(1)}_{k'\bar{j}'}} + 2f_2^{-2}\dfrac{\partial f_1}{\partial \bar{z}^{l}}\dfrac{\partial f_1}{\partial z^{s}}h_{k'\bar{j}'}g^{s\bar{l}}, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\bar{j}'} = 0. \end{cases} \tag{36}$$

where ${K^{(1)}_{k\bar{j}}}^1$ and ${K^{(1)}_{k'\bar{j}'}}^2$ are coefficients of the first Levi-Civita Ricci curvature of g and h, respectively.

Proof. From (9) and (16), we get

$$K^{(1)}_{k\bar{j}} = G^{\gamma\bar{\delta}}K_{k\bar{j}\gamma\bar{\delta}} = G^{s\bar{p}}K_{k\bar{j}s\bar{p}} + G^{s'\bar{p}'}K_{k\bar{j}s'\bar{p}'}. \tag{37}$$

According to (16) and the first equation of proposition 8, we have

$$G^{s\bar{p}}K_{k\bar{j}s\bar{p}} = f_2^{-2}g^{s\bar{p}}\left(f_2^2\overset{1}{K_{k\bar{j}s\bar{p}}} + f_1^{-2}f_2^2h^{i'\bar{l}'}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{s\bar{j}}g_{k\bar{p}}\right) = \overset{1}{K^{(1)}_{k\bar{j}}} + f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{k\bar{j}}h^{i'\bar{l}'}. \tag{38}$$

Similarly, by using (16) and the third equation of proposition 8, we can get

$$G^{s'\bar{p}'}K_{k\bar{j}s'\bar{p}'} = f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}g_{k\bar{j}}h^{s'\bar{l}'}. \tag{39}$$

Replacing the summation index i' on the right side of (38) with s' and then taking it and (39) into (37), we can obtain

$$K^{(1)}_{k\bar{j}} = \overset{1}{K^{(1)}_{k\bar{j}}} + 2f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}g_{k\bar{j}}h^{s'\bar{l}'}. \tag{40}$$

Similarly, we can obtain

$$\begin{aligned} K^{(1)}_{k'\bar{j}'} &= \overset{2}{K^{(1)}_{k'\bar{j}'}} + 2f_2^{-2}\frac{\partial f_1}{\partial \bar{z}^{l}}\frac{\partial f_1}{\partial z^{s}}h_{k'\bar{j}'}g^{s\bar{l}}, \\ K^{(1)}_{k'\bar{j}} &= 0, \\ K^{(1)}_{k\bar{j}'} &= 0. \end{aligned} \tag{41}$$

This completes the proof. □

Theorem 10. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). Then, the Levi-Civita Ricci scalar curvature of G along a nonzero vector $v = (v^i, v^{i'}) \in T^{1,0}_z M$ is given by*

$$\begin{aligned} S_{LC}(v) = {} & f_2^{-2}S_g(v_1) + f_1^{-2}S_h(v_2) + 2f_1^{-2}f_2^{-2}g^{s\bar{l}}\frac{\partial f_1}{\partial \bar{z}^{l}}\frac{\partial f_1}{\partial z^{s}} \\ & + 2f_1^{-2}f_2^{-2}h^{s'\bar{l}'}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}, \end{aligned} \tag{42}$$

where $S_g(v_1)$ and $S_h(v_2)$ are Levi-Civita Ricci scalar curvatures of g and h, respectively.

Proof. According to (11), the Levi-Civita Ricci scalar curvature of G is given by

$$S_{LC} = G^{\alpha\bar{\beta}}K^{(1)}_{\alpha\bar{\beta}} = G^{k\bar{j}}K^{(1)}_{k\bar{j}} + G^{k'\bar{j}'}K^{(1)}_{k'\bar{j}'} + G^{k'\bar{j}}K^{(1)}_{k'\bar{j}} + G^{k\bar{j}'}K^{(1)}_{k\bar{j}'}. \tag{43}$$

Combining (16) and (40), we have

$$\begin{aligned} G^{k\bar{j}}K^{(1)}_{k\bar{j}} &= f_2^{-2}g^{k\bar{j}}\left(\overset{1}{K^{(1)}_{k\bar{j}}} + 2f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}g_{k\bar{j}}h^{s'\bar{l}'}\right) \\ &= f_2^{-2}S_g(v_1) + 2f_1^{-2}f_2^{-2}h^{s'\bar{l}'}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}}. \end{aligned} \tag{44}$$

Similarly, we can get

$$G^{k'\bar{j}'}K^{(1)}_{k'\bar{j}'} = f_1^{-2}S_h(v_2) + 2f_1^{-2}f_2^{-2}g^{s\bar{l}}\frac{\partial f_1}{\partial \bar{z}^{l}}\frac{\partial f_1}{\partial z^{s}}, \tag{45}$$

$$G^{k'\bar{j}}K^{(1)}_{k'\bar{j}} = 0, \tag{46}$$

$$G^{k\bar{j}'}K^{(1)}_{k\bar{j}'} = 0. \tag{47}$$

Taking (44)–(47) into (43), we obtain (42). □

Theorem 11. *Let $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, then $S_{LC}(v) = f_2^{-2}S_g(v_1) + f_1^{-2}S_h(v_2)$.*

Proof. If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, i.e.,

$$\begin{aligned} \frac{\partial f_1}{\partial \bar{z}^l} &= 0, \\ \frac{\partial f_2}{\partial \bar{z}^{l'}} &= 0. \end{aligned} \tag{48}$$

Thus,

$$2f_1^{-2}f_2^{-2}g^{s\bar{l}}\frac{\partial f_1}{\partial \bar{z}^l}\frac{\partial f_1}{\partial z^s} = 0, \tag{49}$$

$$2f_1^{-2}f_2^{-2}h^{s'\bar{l}'}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{s'}} = 0. \tag{50}$$

Substituting (49) into (42), we have $S_{LC}(v) = f_2^{-2}S_g(v_1) + f_1^{-2}S_h(v_2)$. □

4. Levi-Civita Ricci-Flat Doubly Warped Product Hermitian Manifolds

Let (M_1, g) and (M_2, h) be two Levi-Civita Ricci-flat Hermitian manifolds; one may want to know whether the DWP-Hermitian manifold $({}_{f_2}M_1 \times_{f_1} M_2, G)$ is also a Levi-Civita Ricci-flat Hermitian manifold. We shall give an answer to this question in this section.

Theorem 12. *Let $({}_{f_2}M_1 \times_{f_1} M_2, G)$ be a DWP-Hermitian manifold of (M_1, g) and (M_2, h). If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, then $({}_{f_2}M_1 \times_{f_1} M_2, G)$ is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat.*

Proof. If f_1 and f_2 are holomorphic functions on M_1 and M_2, respectively, i.e.,

$$\begin{aligned} \frac{\partial f_1}{\partial \bar{z}^l} &= 0, \\ \frac{\partial f_2}{\partial \bar{z}^{l'}} &= 0. \end{aligned} \tag{51}$$

Taking above equations into the first formula and second formula of (36), we get

$$2f_1^{-2}\frac{\partial f_2}{\partial \bar{z}^{l'}}\frac{\partial f_2}{\partial z^{i'}}g_{k\bar{j}}h^{i'\bar{l}'} = 0, \tag{52}$$

$$2f_2^{-2}\frac{\partial f_1}{\partial \bar{z}^l}\frac{\partial f_1}{\partial z^s}h_{k'\bar{j}'}g^{s\bar{l}} = 0. \tag{53}$$

Firstly, we assume $({}_{f_2}M_1 \times_{f_1} M_2, G)$ be Levi-Civita Ricci-flat; using Definition 2 and (36), we have

$$\begin{cases} K^{(1)}_{k\bar{j}} = \overset{1}{K}{}^{(1)}_{k\bar{j}} + 2f_1^{-2}\dfrac{\partial f_2}{\partial \bar{z}^{l'}}\dfrac{\partial f_2}{\partial z^{i'}}g_{k\bar{j}}h^{i'\bar{l}'} = 0, \\ K^{(1)}_{k'\bar{j}'} = \overset{2}{K}{}^{(1)}_{k'\bar{j}'} + 2f_2^{-2}\dfrac{\partial f_1}{\partial \bar{z}^{l}}\dfrac{\partial f_1}{\partial z^{s}}h_{k'\bar{j}'}g^{s\bar{l}} = 0, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\bar{j}'} = 0. \end{cases} \tag{54}$$

Substituting (52) and (53) into the first formula and second formula of (54), respectively, we get

$$\begin{cases} K^{(1)}_{k\bar{j}} = \overset{1}{K}{}^{(1)}_{k\bar{j}} = 0, \\ K^{(1)}_{k'\bar{j}'} = \overset{2}{K}{}^{(1)}_{k'\bar{j}'} = 0, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\bar{j}'} = 0. \end{cases} \tag{55}$$

Obviously,

$$\begin{aligned} \overset{1}{K}{}^{(1)}_{k\bar{j}} &= 0, \\ \overset{2}{K}{}^{(1)}_{k'\bar{j}'} &= 0. \end{aligned} \tag{56}$$

According to Definition 2, these mean that (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat.

Conversely, we assume (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat; according to Definition 2, we know that

$$\overset{1}{K}{}^{(1)}_{k\bar{j}} = 0, \tag{57}$$

$$\overset{2}{K}{}^{(1)}_{k'\bar{j}'} = 0. \tag{58}$$

Since f_1 and f_2 are holomorphic, thus (52) and (53) are established. Then, taking (52), (53), (57), and (58) into (36), we obtain

$$\begin{cases} K^{(1)}_{k\bar{j}} = 0, \\ K^{(1)}_{k'\bar{j}'} = 0, \\ K^{(1)}_{k'\bar{j}} = 0, \\ K^{(1)}_{k\bar{j}'} = 0. \end{cases} \tag{59}$$

By Definition 2, (59) indicates that $({}_{f_2}M_1 \times {}_{f_1}M_2, G)$ is Levi-Civita Ricci-flat. □

Notation 13. Theorem 12 implies that when warped functions to be holomorphic, then the DWP-Hermitian manifold is a Levi-Civita Ricci-flat Hermitian manifold if and only if its component manifolds are Levi-Civita Ricci-flat. Thus, this theorem provides us an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

5. Conclusions

In this paper, we derived formulae of Levi-Civita connection, Levi-Civita curvature, the first Levi-Civita Ricci curvature, and Levi-Civita scalar curvature of the DWP-Hermitian manifold and proved that if the warped function f_1 and f_2 are holomorphic, then the DWP-Hermitian manifold is Levi-Civita Ricci-flat if and only if (M_1, g) and (M_2, h) are Levi-Civita Ricci-flat manifolds. Thus, we gave an effective way to construct Levi-Civita Ricci-flat DWP-Hermitian manifold.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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