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Superharmonic Solutions of Order (2,3) to a MEMS Governed the Motion of a TM-AFM Cantilever

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Authors' contributions

This work was carried out in collaboration between all authors. All the authors worked in co-operation to present the work featured in this paper. All authors read and approved the final manuscript.

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ABSTRACT

The superharmonic solutions of a weakly nonlinear second order differential equation governed the dynamic behavior of a microcantilever based on TM (Tapping mode) AFM (Atomic force microscopy) are studied analytically by applying the method of multiple scales (MMS). The modulation equations of the amplitude and the phase are obtained. Steady state solutions, frequency response equations, the peak amplitudes with their locations and the approximate analytical expressions are obtained. The stability of the steady state solutions is given. Numerical solutions of the frequency response equations and their stability conditions are carried out for different values of the parameters in the equation. Results are presented in groups of figures. Finally, discussion and conclusion are given.

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1 INTRODUCTION

Atomic force microscopy (AFM) is a fundamental tool in modern nanoscience. It has many applications in manipulation of carbon nanotubes, nanolithography, organic molecules, data-storage technologies and semiconductor devices. The idea of AFM is based on the monitoring and control of the dynamics of a sharp probe tip which is attached to a flexible microcantilever and interacts with the sample surface. The atomic force microscope (AFM) is designed to exploit this level of sensitivity. The ability to accurately measure material properties at nanometer length scales is a critical challenge in the design, manufacture of many emerging materials and systems. It has special important in nanoscale mechanical properties including elasticity, friction, plasticity and wear, as they can significantly influence macro-scale behavior. In particular, several dynamic techniques where the AFM tip, cantilever base, or the samples are subjected to a periodic excitation have been used for many applications. Different modes of AFM operation may be set up according to the frequency and amplitude of the vibration applied and the AFM component that is applied to it [1, 2, 3, 4, 5].

The mathematically study of most classical dynamical systems and nonclassical dynamical systems (Micro and Nano-electro-mechanical systems (MEMS/NENS)) yields to nonlinear second order ordinary differential equations or a set of a nonlinear coupled second order ordinary differential equations. So there exist significant efforts to study the different types of periodic solutions of these ordinary differential equations (harmonic, sub, super, sub-super, super-sub and combinations harmonic solutions) by using the perturbation technique [6, 7, 8]. Elnaggar and El-Bassiouny [9] investigated superharmonic solution of self-excited two coupled second order systems to multi-frequency excitations. In the particular case of a superharmonic solution, Piccirillo et al. [10] studied the dynamics of the shape memory oscillator. They obtained an approximate solution to the governing equations of motion. Shooshtari and Pasha Zanoosi [11] represented the superharmonic solution of second order weakly nonlinear differential equation that represents the vibration of a mass grounded system which includes two linear and nonlinear springs in series. Caruntu and Knechty [12] investigated analytically superharmonic solution of the governing equation of electrostatically actuated microresonators. The superharmonic solution of a forced single degree of freedom (SDOF) nonlinear system was represented by Elnaggar et al. [13]. Ji and Zhang [14] investigated the superharmonic solution of a weakly nonlinear oscillator having cubic nonlinearity. The study of a superharmonic solution of the governing equation of a nonlinear cantilever beam with tip mass subject to an axial force and electrostatic excitation are represented by Kim et al. [15]. Elnaggar and Khalil [16] discussed the superharmonic solution for nonlinear system with two distinct time-delays under an external excitation. The superharmonic solution of a modified Duffing-Van der Pol equation subjected to weakly nonlinear parametric and external excitations represented by Elnaggar *et al.* [17, 18]. Wen-Ming Zhang et al. [19] studied the dynamic behavior of a micro cantilever based TM-AFM with squeeze film damping effects using numerical simulation. Elnaggar et al. [20] investigated the harmonic solution of a weakly non-linear second order differential equation governed the dynamic behavior of a micro cantilever based on TM (Tapping mode) AFM (Atomic force microscopy).

The object of this work is to study superharmonic solutions of order $m(m = 2, 3)$ (i.e. periodic solution has as its least period $1/m$ of the period of the external excitation) of a weakly nonlinear second order differential equation governed the motion of micro cantilever based TM-AFMs with squeeze film damping by using the perturbation technique (Method of multiple scales) [7, 8]. The modulation equations of the amplitude and the phase are determined. Steady state solutions, frequency response equations and stability analysis are given. Peak amplitude and its localization are determined. Numerical solutions for the frequency response equation and the stability conditions are carried out. Results are presented in group of figures in which solid (dashed) curves represented stable (unstable) superharmonic solutions. Finally, discussion and conclusion are given.

2 FORMULATION OF THE PROBLEM AND PERTURBATION ANALYSIS

Consider the following nonlinear second order differential equation [19]

$$u'' + \zeta u' + u + \beta u^3 = -\frac{d}{(\alpha+u)^2} + \frac{d\Sigma^6}{30(\alpha+u)^8} + \epsilon\left(f\text{Cos}\Omega t - \frac{\eta}{(\alpha+u)^3}u'\right) \quad (2.1)$$

Where α represent the equilibrium coefficient ratio, β mean the cubic stiffness ratio, d is constant, Σ mean the the material property parameter, ζ is the coefficient of damping term, η represent the squeeze film damping, Ω and f are the frequency and the amplitude of the eternal excitation of the system.

By using Taylor expansion and retained only three terms of expansion. For applying perturbation technique, we introduce a small parameter $\epsilon << 1$ in the nonlinear terms and taking the amplitude of the excitation force of *O(1)*, then we get the following weakly nonlinear second order differential equation:

$$u'' + \omega_0^2 u + \epsilon(2\mu u' - \alpha_3 u^2 + \beta u^3 - \alpha_5 u u' + \alpha_6 u^2 u') = \alpha_1 + f\text{Cos}\Omega t \quad (2.2)$$

Where $\omega_0^2 = 1-\alpha_2, 2\mu = \zeta + \alpha_4 = \zeta + \frac{\eta}{\alpha^3}, \alpha_1 = \frac{d\Sigma^6}{30\alpha^8} - \frac{d}{\alpha^2}, \alpha_2 = \frac{2d}{\alpha^3} - \frac{4d\Sigma^6}{15\alpha^9}, \alpha_3 = \frac{6d\Sigma^6}{5\alpha^{10}} - \frac{3d}{\alpha^4}$, $\alpha_4 = \frac{\eta}{\alpha^3}, \alpha_5 = \frac{3\eta}{\alpha^4}$ and $\alpha_6 = \frac{6\eta}{\alpha^5}$.

An approximate solution of equation (2.2) can be obtained by using the method of multiple scales (Nayfeh [7, 8]), we get a first order uniform solution of equation (2.2) in the form

$$u(t;\epsilon) = u_0(T_0,T_1) + \epsilon u_1(T_0,T_1) + ..., \quad (2.3)$$

where $T_0 = t$ is the first scale associated with changes occurring at the frequencies ω_0 and Ω, and $T_1 = \epsilon t$ is a slow scale associated with modulations in the amplitude. In terms of T_1, the time derivatives become

$$\frac{d}{dt} = D_0 + \epsilon D_1 + ... \qquad \&$$

$$\frac{d^2}{dt^2} = D_0^2 + 2\epsilon D_0 D_1 + ... \quad (2.4)$$

where $D_n = \frac{\partial}{\partial T_n}$. Substituting equations (2.3) and (2.4) into equation (2.2) and equating coefficients of like powers of ϵ to zero, we obtain a set of linear differential equations

$$D_0^2 u_0 + \omega_0^2 u_0 = \alpha_1 + f\text{Cos}\Omega t \quad (2.5)$$

$$D_0^2 u_1 + \omega_0^2 u_1 = -2\mu D_0 u_0 - 2D_1 D_0 u_0 - \beta u_0^3 + \alpha_3 u_0^2 + \alpha_5 u_0 D_0 u_0 - \alpha_6 u_0^2 D_0 u_0 \quad (2.6)$$

Solving the equation (2.5) for $u_0(T_0,T_1)$, we have

$$u_0(T_0,T_1) = A(T_1)e^{i\omega_0 T_0} + \bar{A}(T_1)e^{-i\omega_0 T_0} + \kappa + \Lambda\text{Cos}\Omega T_0 \quad (2.7)$$

where $i^2 = 1$, $\bar{A}$ is the complex conjugate of A, $\kappa = \frac{\alpha_1}{\omega_0^2}$ *and* $\Lambda = \frac{f}{\omega_0^2-\Omega^2}$. Then equation (2.6) becomes,

$$D_0{}^2 u_1 + \omega_0^2 u_1 = -\frac{1}{2}e^{iT_0\omega_0}(A(i\omega_0(\alpha_6(2A\bar{A} + 2\kappa^2 + \Lambda^2) - 2\alpha_5\kappa + 4\mu) + 6A\beta\bar{A} - 4\alpha_3\kappa + 6\beta\kappa^2 + 3\beta\Lambda^2) + 4i\omega_0 A') + 2\alpha_3 A\bar{A} - 6A\beta\kappa\bar{A} - \frac{1}{8}\Lambda e^{iT_0\Omega}(i\alpha_6\Omega(8A\bar{A} + 4\kappa^2 + \Lambda^2)$$

$$
\begin{aligned}
&+ 24A\beta\bar{A} - 4i\alpha_5\kappa\Omega - 8\alpha_3\kappa + 12\beta\kappa^2 + 3\beta\Lambda^2 + 8i\Omega\mu) \\
&- \frac{1}{2}\Lambda\bar{A}e^{iT_0(\Omega-\omega_0)}\left(i\left(\alpha_5 - 2\alpha_6\kappa\right)\left(\omega_0 - \Omega\right) - 2\alpha_3 + 6\beta\kappa\right) \\
&- \frac{1}{4}\Lambda^2\bar{A}e^{iT_0(2\Omega-\omega_0)}\left(3\beta + i\alpha_6\left(2\Omega - \omega_0\right)\right) \\
&- \frac{1}{2}\Lambda\bar{A}^2e^{iT_0(\Omega-2\omega_0)}\left(3\beta + i\alpha_6\left(\Omega - 2\omega_0\right)\right) \\
&+ \alpha_3\kappa^2 + \frac{\alpha_3\Lambda^2}{2} - \beta\kappa^3 - \frac{3}{2}\beta\kappa\Lambda^2 \\
&- \frac{1}{4}\Lambda^2e^{2iT_0\Omega}\left(-i\Omega\left(\alpha_5 - 2\alpha_6\kappa\right) - \alpha_3 + 3\beta\kappa\right) \\
&- \frac{1}{8}\Lambda^3e^{3iT_0\Omega}\left(\beta + i\alpha_6\Omega\right) + NST. + c.c.
\end{aligned}
\tag{2.8}
$$

where $NST.$ denotes the terms does not produce secular terms and $c.c.$ denotes the complex conjugate.

3 SUPERHARMONIC SOLUTIONS

In this article, we concerned our attention to two types of periodic solutions (superharmonic solutions of order m, $m = 2, 3$)

3.1 Superharmonic Solution of Order 2

To obtain the superharmonic solution of order 2, *i.e.* periodic solution has its least $\frac{1}{2}$ of the period of external excitation; $2\Omega \approx \omega_0$ or

$$2\Omega = \omega_0 + \epsilon\sigma \tag{3.1}$$

Where σ is a detuning parameter. Hence the excitation force can be written

$$f\cos\Omega t = f\cos\frac{1}{2}\left(\omega_0 T_0 + \sigma \mathsf{T}_1\right) \tag{3.2}$$

Eliminating the secular terms(coefficient of $e^{iT_0\omega_0}$) from the equation (2.8) yields

$$
\begin{aligned}
&- 2A(i\omega_0\left(\alpha_6\left(2A\bar{A} + 2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa + 4\mu\right) + 6A\beta\bar{A} - 4\alpha_3\kappa \\
&+ 6\beta\kappa^2 + 3\beta\Lambda^2) - 8i\omega_0 A' + \Lambda^2 e^{i\epsilon\sigma T_0}(i\Omega\left(\alpha_5 - 2\alpha_6\kappa\right) \\
&+ \alpha_3 - 3\beta\kappa) = 0
\end{aligned}
\tag{3.3}
$$

Putting A in polar form as

$$A = \frac{1}{2}a(T_1)e^{i\beta_1(T_1)} \tag{3.4}$$

Where a and β_1 are real functions of T_1. By substituting equation (3.4) into equation (3.3) and separating the real and the imaginary parts, we obtain a set of autonomous differential equations that govern the amplitude $a(T_1)$ and the phase $\gamma(T_1)$ which are known as the modulation equations.

$$
\begin{aligned}
a' &= \frac{2\Lambda^2(J_1\sin(\gamma) + \Omega J_2\cos(\gamma)) - a\omega_0(\alpha_6 J_3 - 4\alpha_5\kappa + 8\mu)}{8\omega_0} \\
a\gamma' &= \frac{a(-3\beta J_3 + 8\alpha_3\kappa + 8\sigma\omega_0) + 2\Lambda^2 J_1\cos(\gamma) - 2\Omega\Lambda^2 J_2\sin(\gamma)}{8\omega_0}
\end{aligned}
\tag{3.5}
$$

where $\gamma = \sigma \mathsf{T}_1 - \beta_1$, $J_1 = (\alpha_3 - 3\beta\kappa)$, $J_2 = (\alpha_5 - 2\alpha_6\kappa)$ and $J_3 = \left(a^2 + 4\kappa^2 + 2\Lambda^2\right)$.

Therefore for the approximate analytical expression of the superharmonic solution of order 2 is

$$u = a\cos[2\Omega t - \gamma] + \frac{f}{\omega_0^2 - \Omega^2}\cos[\Omega t] + \frac{\alpha_1}{\omega_0^2} + o(\epsilon) \tag{3.6}$$

Where γ and a are given by (3.5).

To obtain the steady state solutions, putting $a' = \gamma' = 0$ in system (3.5), then we have a set of algebraic equations for amplitude a and phase γ of the steady state solutions (superharmonic solution of order 2).

$$\begin{aligned} -3a\beta J_3 + 8a\alpha_3\kappa + 2\Lambda^2\cos(\gamma)J_1 + 8a\sigma\omega_0 &= 2\Omega\Lambda^2 J_2\sin(\gamma) \\ a\omega_0\left(\alpha_6 J_3 - 4\alpha_5\kappa + 8\mu\right) - 2\Lambda^2\sin(\gamma)J_1 &= 2\Omega\Lambda^2 J_2\cos(\gamma) \end{aligned} \tag{3.7}$$

Squaring both equations in system (3.7) and adding, we get the *frequency response* equation in the form

$$\begin{aligned} & a^6\left(\alpha_6^2\omega_0^2 + 9\beta^2\right) + 4a^4(-12\alpha_3\beta\kappa + \alpha_6\omega_0^2\left(\alpha_6\left(2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa + 4\mu\right) \\ & + 9\beta^2\left(2\kappa^2 + \Lambda^2\right) - 12\beta\sigma\omega_0) - 4a^3\Omega\Lambda^2 J_2\left(\alpha_6\omega_0\cos(\gamma) - 3\beta\sin(\gamma)\right) \\ & + 4a^2(-8\sigma\omega_0\left(3\beta\left(2\kappa^2 + \Lambda^2\right) - 4\alpha_3\kappa\right) - 8\alpha_3\kappa(3\beta\left(2\kappa^2 + \Lambda^2\right) \\ & - 2\alpha_3\kappa) + \omega_0^2((\alpha_6\left(2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa)\left(\alpha_6\left(2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa + 8\mu\right) \\ & + 16(\mu^2 + \sigma^2)) + 9\beta^2\left(2\kappa^2 + \Lambda^2\right)^2) \\ & + a(8\Omega\Lambda^2\left(\alpha_5 - 2\alpha_6\kappa\right)\sin(\gamma)\left(-4\alpha_3\kappa + 3\beta\left(2\kappa^2 + \Lambda^2\right) - 4\sigma\omega_0\right) \\ & - 8\Omega\Lambda^2\omega_0\left(\alpha_5 - 2\alpha_6\kappa\right)\cos(\gamma)\left(\alpha_6\left(2\kappa^2 + \Lambda^2\right) - 2\alpha_5\kappa + 4\mu\right)) \\ & - 4\Lambda^4\left(\left(\alpha_3 - 3\beta\kappa\right)^2 - \Omega^2\left(\alpha_5 - 2\alpha_6\kappa\right)^2\right) = 0 \end{aligned} \tag{3.8}$$

solving equation (3.8) for σ, we obtain

$$\begin{aligned} \sigma = & \frac{3a\beta\left(a^2 + 4\kappa^2 + 2\Lambda^2\right) - 8a\alpha_3\kappa + 2\Omega\Lambda^2\left(\alpha_5 - 2\alpha_6\kappa\right)\sin(\gamma)}{8a\omega_0} \\ & \pm \frac{1}{8a^2\omega_0^2}\sqrt{a^2\omega_0^2\left(4J_1^2\Lambda^4 - \left(a\omega_0\left(-4\alpha_5\kappa + \alpha_6 J_3 + 8\mu\right) - 2J_2\Omega\Lambda^2\cos(\gamma)\right)^2\right)} \end{aligned} \tag{3.9}$$

The peak amplitude would be verifying the following equation

$$4J_1^2\Lambda^4 - \left(a_p\omega_0\left(-4\alpha_5\kappa + \alpha_6 J_3 + 8\mu\right) - 2J_2\Omega\Lambda^2\cos(\gamma)\right)^2 = 0 \tag{3.10}$$

Then the corresponding value of σ is given from

$$\sigma_p = \frac{3a_p\beta\left(a_p^2 + 4\kappa^2 + 2\Lambda^2\right) - 8a_p\alpha_3\kappa + 2\Omega\Lambda^2 J_2\sin(\gamma)}{8a_p\omega_0} \tag{3.11}$$

The stability of superharmonic solutions of order 2 can be examined by introducing a small perturbation to the steady state solutions i. e. putting

$$a = a_0 + a_1 \tag{3.12}$$

$$\gamma = \gamma_0 + \gamma_1 \tag{3.13}$$

Where a_0 and γ_0 represent the steady state solution a_1 and γ_1 represent the perturbation. Substituting equations (3.12) and (3.13) into system (3.5), by using the steady state condition and keeping linear terms, one obtains

$$\begin{aligned} a_1' &= \frac{\left(-\omega_0\left(3a_0^2\alpha_6 + 4\alpha_6\kappa^2 - 4\alpha_5\kappa + 2\alpha_6\Lambda^2 + 8\mu\right)a_1 - J_4\gamma_1\right)}{8\omega_0} \\ \gamma_1' &= -\frac{J_5 a_1 + \omega_0\left(a_0\left(\alpha_6\left(a_0^2 + 4\kappa^2 + 2\Lambda^2\right) - 4\alpha_5\kappa + 8\mu\right)\right)\gamma_1}{8a_0\omega_0} \end{aligned} \tag{3.14}$$

where $J_4 = 8a_0\sigma\omega_0 - a_0\left(3\beta\left(a_0^2+4\kappa^2+2\Lambda^2\right)-8\alpha_3\kappa\right)$ and $J_5 = 9a_0^2\beta - 8\alpha_3\kappa + 6\beta\left(2\kappa^2+\Lambda^2\right) - 8\sigma\omega_0$. Substituting $a_1 = \Gamma_1 e^{\theta T_1}$ and $\gamma_1 = \Gamma_2 e^{\theta T_1}$ into system (3.14). We get

$$\begin{aligned} J_5\Gamma_1 + a_0\omega_0\left(\alpha_6\left(a_0^2+4\kappa^2+2\Lambda^2\right)-4\alpha_5\kappa+8(\theta+\mu)\right)\Gamma_2 &= 0 \\ \left(\omega_0\left(\alpha_6\left(3a_0^2+4\kappa^2+2\Lambda^2\right)-4\alpha_5\kappa\right)+8\omega_0(\theta+\mu)\right)\Gamma_1 + J_4\Gamma_2 &= 0 \end{aligned} \tag{3.15}$$

For obtaining the nontrivial solution the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ.

$$\theta = \frac{1}{8}\left(-2\alpha_6\left(a_0^2+2\kappa^2+\Lambda^2\right)+4\alpha_5\kappa-8\mu \pm \frac{\sqrt{\alpha_6^2 a_0^6\omega_0^4 + a_0 J_4 J_5 \omega_0^2}}{a_0\omega_0^2}\right) \tag{3.16}$$

The stability of the superharmonic solution can be examined by evaluating the sign of the real part of the eigenvalues. Consequently, a solution is stable if and only if the real parts of both eigenvalues of equation (3.16) are less than zero.

3.2 Superharmonic Solution of Order 3

To obtain the superharmonic solution of order 3, (i.e. periodic solution has its least period $1/3$ the period of external excitation), hence $3\Omega \approx \omega_0$ or

$$3\Omega = \omega_0 + \epsilon\sigma \tag{3.17}$$

where is σ a detuning parameter. Hence the excitation can be written as

$$f\cos(\Omega t) = f\cos\left(\frac{1}{3}(\omega_0 T_0 + \sigma T_1)\right) \tag{3.18}$$

By eliminating the secular terms from the equation (2.8), we get

$$\begin{aligned} &12A(-i\omega_0\left(\alpha_6\left(2A\bar{A}+2\kappa^2+\Lambda^2\right)-2\alpha_5\kappa+4\mu\right)-6A\beta\bar{A}+4\alpha_3\kappa \\ &-6\beta\kappa^2-3\beta\Lambda^2)-48i\omega_0 A' - \Lambda^3 e^{i\epsilon\sigma T_0}\left(3\beta+i\alpha_6\Omega\right)=0 \end{aligned} \tag{3.19}$$

Using the polar form $A = \frac{1}{2}a(T_1)e^{i\beta_1(T_1)}$ into the equation (3.19) and separating real and imaginary parts, we obtain a set of autonomous differential equations that govern the amplitude $a(T_1)$ and the phase $\gamma(T_1)$ which known the modulation equations:

$$\begin{aligned} a' &= -\frac{a\omega_0\left(\alpha_6\left(a^2+4\kappa^2+2\Lambda^2\right)-4\alpha_5\kappa+8\mu\right)+\Lambda^3(\alpha_6\Omega\cos(\gamma)+\beta\sin(\gamma))}{8\omega_0} \\ a\gamma' &= \frac{-\beta\left(3a\left(a^2+4\kappa^2+2\Lambda^2\right)+\Lambda^3\cos(\gamma)\right)+8a\alpha_3\kappa+\alpha_6\Lambda^3\Omega\sin(\gamma)+8a\sigma\omega_0}{8\omega_0} \end{aligned} \tag{3.20}$$

where $\gamma = \sigma T_1 - \beta_1$.

Therefore for the approximate analytical expression of the superharmonic solution of order 3 is

$$u = a\cos[3\Omega t - \gamma] + \frac{f}{\omega_0^2 - \Omega^2}\cos[\Omega t] + \frac{\alpha_1}{\omega_0^2} + o(\epsilon) \tag{3.21}$$

Where a and γ are given by (3.20).

To obtain the steady state solutions, putting $a' = \gamma' = 0$ in system (3.20), then we have a set of algebraic equations for amplitude a and phase γ of the steady state solutions (superharmonic solution of order 3).

$$\begin{aligned} 4a\omega_0\left(\alpha_5\kappa-2\mu\right)-\alpha_6\left(a\omega_0\left(a^2+4\kappa^2+2\Lambda^2\right)+\Lambda^3\Omega\cos(\gamma)\right) &= \beta\Lambda^3\sin(\gamma) \\ -3a\beta\left(a^2+4\kappa^2+2\Lambda^2\right)+8a\alpha_3\kappa+\alpha_6\Lambda^3\Omega\sin(\gamma)+8a\sigma\omega_0 &= \beta\Lambda^3\cos(\gamma) \end{aligned} \tag{3.22}$$

Squaring and adding both equations in system (3.22), we get the *frequency response* equation

$$\begin{aligned} & a^6\left(\alpha_6^2\omega_0^2+9\beta^2\right)+4a^4(-12\alpha_3\beta\kappa+\alpha_6\omega_0^2\left(\alpha_6\left(2\kappa^2+\Lambda^2\right)-2\alpha_5\kappa+4\mu\right) \\ & +9\beta^2\left(2\kappa^2+\Lambda^2\right)-12\beta\sigma\omega_0)+2\alpha_6a^3\Lambda^3\left(\alpha_6\omega_0\Omega\cos(\gamma)-3\beta\Omega\sin(\gamma)\right) \\ & +4a^2(-8\sigma\omega_0\left(3\beta\left(2\kappa^2+\Lambda^2\right)-4\alpha_3\kappa\right)-8\alpha_3\kappa\left(3\beta\left(2\kappa^2+\Lambda^2\right)-2\alpha_3\kappa\right) \\ & +\omega_0^2((\alpha_6\left(2\kappa^2+\Lambda^2\right)-2\alpha_5\kappa)\left(\alpha_6\left(2\kappa^2+\Lambda^2\right)-2\alpha_5\kappa+8\mu\right) \\ & +16\left(\mu^2+\sigma^2\right))+9\beta^2\left(2\kappa^2+\Lambda^2\right)^2)+4\alpha_6a\Lambda^3(\Omega\sin(\gamma)(4\alpha_3\kappa \\ & -3\beta\left(2\kappa^2+\Lambda^2\right)+4\sigma\omega_0)+\omega_0\Omega\cos(\gamma)(\alpha_6\left(2\kappa^2+\Lambda^2\right) \\ & -2\alpha_5\kappa+4\mu))+\alpha_6^2\Lambda^6\Omega^2-\beta^2\Lambda^6=0 \end{aligned} \tag{3.23}$$

Solving equation (3.23) for σ, we obtain

$$\begin{aligned} \sigma = & \frac{1}{8a^2\omega_0^2}[a\omega_0\left(3a\beta\left(a^2+4\kappa^2+2\Lambda^2\right)-8a\alpha_3\kappa+\alpha_6\Lambda^3(-\Omega)\sin(\gamma)\right) \\ & \pm\sqrt{a^2\omega_0^2\left(\beta^2\Lambda^6-\left(a\omega_0\left(\alpha_6\left(a^2+4\kappa^2+2\Lambda^2\right)-4\alpha_5\kappa+8\mu\right)+\alpha_6\Lambda^3\Omega\cos(\gamma)\right)^2\right)}] \end{aligned} \tag{3.24}$$

The peak amplitude would be verifying the following equation

$$\beta^2\Lambda^6-\left(a_p\omega_0\left(\alpha_6\left(a_p^2+4\kappa^2+2\Lambda^2\right)-4\alpha_5\kappa+8\mu\right)+\alpha_6\Lambda^3\Omega\cos(\gamma)\right)^2=0 \tag{3.25}$$

Then the corresponding value of σ is given from

$$\sigma_p=\frac{3a_p\beta\left(a_p^2+4\kappa^2+2\Lambda^2\right)-8a_p\alpha_3\kappa-\alpha_6\Lambda^3\Omega\sin(\gamma)}{8a_p\omega_0} \tag{3.26}$$

The stability of superharmonic solutions of order 3 can be examined by introducing a small perturbation to the steady state solutions i. e. putting

$$a=a_0+a_1 \tag{3.27}$$

$$\gamma=\gamma_0+\gamma_1 \tag{3.28}$$

Where a_0 and γ_0 represent the steady state solution a_1 and γ_1 represent the perturbation. Substituting equations (3.27) and (3.28) into system (3.20) by using the steady state condition and keeping linear terms, one obtains

$$\begin{aligned} a_1' &= -\tfrac{1}{8}\left(\alpha_6J_6-4\alpha_5\kappa+8\mu\right)a_1-\tfrac{J_7}{8\omega_0}\gamma_1 \\ \gamma_1' &= \tfrac{(-J_9)}{8a_0\omega_0}a_1-\tfrac{1}{8}J_8\gamma_1 \end{aligned} \tag{3.29}$$

Where

$J_6=3a_0^2+4\kappa^2+2\Lambda^2$, $J_7=8a_0\sigma\omega_0-a_0\left(3a_0^2\beta-8\alpha_3\kappa+6\beta\left(2\kappa^2+\Lambda^2\right)\right)$,
$J_8=a_0^2\alpha_6+4\alpha_6\kappa^2-4\alpha_5\kappa+2\alpha_6\Lambda^2+8\mu$ and $J_9=9a_0^2\beta-8\alpha_3\kappa+12\beta\kappa^2+6\beta\Lambda^2-8\sigma\omega_0$
Substituting $a_1=\Gamma_1e^{\theta T_1}$ and $\gamma_1=\Gamma_2e^{\theta T_1}$ into system (3.29). We get

$$\begin{aligned} \omega_0\left(\alpha_6\left(3a_0^2+4\kappa^2+2\Lambda^2\right)-4\alpha_5\kappa+8(\theta+\mu)\right)\Gamma_1+J_7\Gamma_2 &= 0 \\ J_9\Gamma_1+a_0\omega_0\left(8\theta+J_8\right)\Gamma_2 &= 0 \end{aligned} \tag{3.30}$$

For obtaining the nontrivial solution the determinant of the coefficient matrix for Γ_1 and Γ_2 must vanish, which leads to a quadratic equation for the eigenvalue θ.

$$\begin{aligned} \theta = & \frac{(-\alpha_6J_6+4\alpha_5\kappa-J_8-8\mu)}{16}) \\ & \pm\frac{1}{16a_0\omega_0^2}\sqrt{a_0\omega_0^2\left(a_0\omega_0^2\left(-\alpha_6J_6+4\alpha_5\kappa+J_8-8\mu\right)^2+4J_7J_9\right)} \end{aligned} \tag{3.31}$$

The stability of the superharmonic solution can be examined by evaluating the sign of the real part of the eigenvalues. Hence, a solution is stable if and only if the real parts of both eigenvalues of equation (3.31) are less than zero.

4 NUMERICAL RESULTS WITH DISCUSSION

This section presents numerical results in the form of frequency response curves obtained by solving the frequency response equations(3.8), (3.23) and stability conditions (3.16), (3.31). The numerical results are plotted in groups of Figs. (1-7) and (8-14), which represent the variation of the amplitude a with the detuning parameter σ for a given values of the other parameters where the solid lines represent stable solutions and the dashed lines represent unstable solutions.

Figs. (1-7) represent the frequency response curves for the superharmonic solution of order 2 for certain values of the parameters $\alpha = 2, \beta = 1, \eta = 0.0635, f = 0.5, \Sigma = 0.3, \zeta = 0.01, d = 4/27$ and $\gamma = 90$.

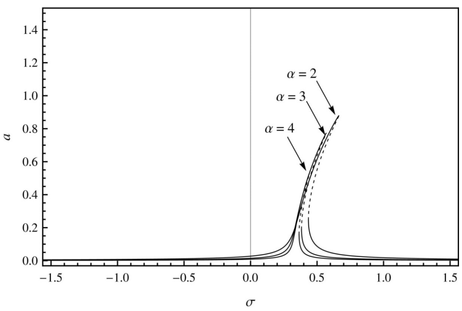

Fig. 1. The frequency response curves for different values of μ

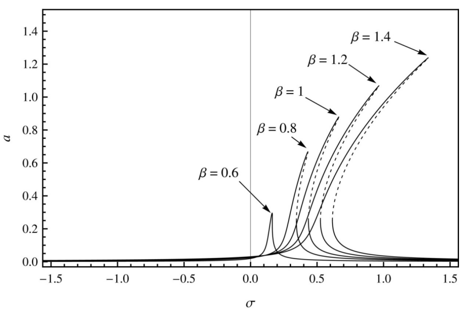

Fig. 2. The frequency response curves for different values of β

Fig.1 shows the variation of the amplitude of the steady state solutions for different values of the equilibrium coefficient ratio α. It can be seen from this figure by decreasing α, the curves bent to right of σ axis, their exist three solutions two stable and one unstable and jump phenomena. From Fig. 2, we note that for the cubic stiffness ratio $\beta = 0.6$, we have one stable solution. By increasing β, the curves bent to the right hand side (R.H.S) and we have three solutions, two stable and one unstable and then jump phenomena for certain values of σ. From Fig. 3, by decreasing the squeeze film damping η, we have three solutions; two stable and one unstable, jump phenomena for certain values of σ and the inclination in the R.H.S. Fig.4 shows that for small values of the external force $f = 0.16$, we have one symmetric stable solution. By increasing f, we have multi-valued solutions for a certain value of σ, two stable solutions and one unstable solution and jump phenomena and the bend in the R.H.S. From Fig. 5, we observe that by increasing the material property parameter Σ, the curves bent to the right of the σ axis and their exist three solutions; two stable, one unstable and jump phenomena. Fig. 6 shows that by decreasing the damping parameter ζ, we have multi-valued solutions for a certain value of σ, two stable solutions and one unstable solution and jump phenomena and the bend in the R.H.S. From Fig. 7 for small values of $d = 1/28$, we have one symmetric stable solution. By increasing d, we have multi-valued solutions for a certain value of σ, two stable solutions and one unstable solution and jump phenomena and the bend in the R.H.S.

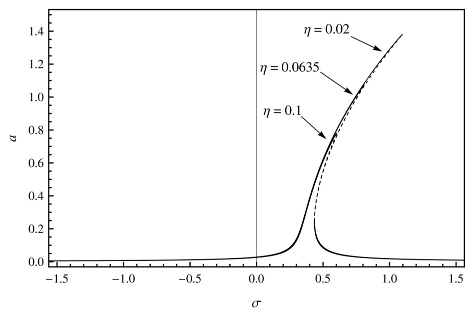

Fig. 3. The frequency response curves for different values of η

Figs. (8-14) represent the frequency response curves for the superharmonic solution of order 3 for certain values of the parameters $\alpha = 2, \beta = 1, \eta = 0.0635, f = 0.5, \Sigma = 0.3, \zeta = 0.01, d = 4/27$ and $\gamma = 90$.

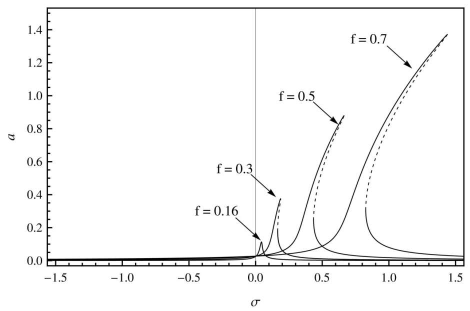

Fig. 4. The frequency response curves for different values of f

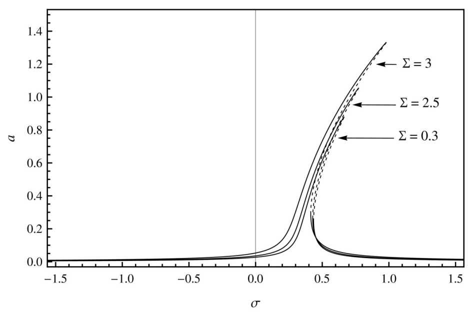

Fig. 5. The frequency response curves for different values of Σ

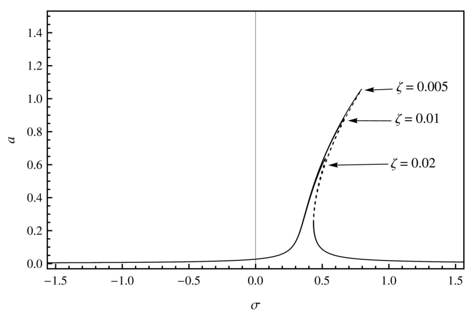

Fig. 6. The frequency response curves for different values of ζ

Fig. 8 shows the variation of the amplitude of the steady state solutions for different values of the equilibrium coefficient ratio α. It can be seen from this figure for $\alpha = 0.9$, we have one stable solution. By increasing α, the curves bent to right of σ axis, their exist three solutions two stable and one unstable and jump phenomena. From Fig. 9, we note that for the cubic stiffness ratio $\beta = 0.2$, we have one stable symmetric solution. By increasing β, the curves bent to the right hand side (R.H.S) and we have three solutions, two stable and one unstable and then jump phenomena for certain values of σ. From Fig. 10, by decreasing the squeeze film damping η, we have three solutions; two stable and one unstable, jump phenomena for certain values of σ and the inclination in the R.H.S. Fig. 11 shows that for small values of the external force $f = 0.1$, we have one stable solution. By increasing f, we have multi-valued solutions for a certain value of σ, two stable solutions and one unstable solution and jump phenomena and the bend in the R.H.S. From Fig. 12, we observe that by decreasing the material property parameter Σ, the curves bent to the right of the σ axis and their exist three solutions two stable, one unstable and jump phenomena. Fig. 13 shows that by reducing the damping parameter ζ, we have multi-valued solutions for a certain value of σ; two stable solutions, one unstable solution, jump phenomena and the bend in the R.H.S. From Fig. 14, we observe that by increasing d, we have multi-valued solutions for a certain value of σ; two stable solutions and one unstable solution, jump phenomena and the bend in the R.H.S.

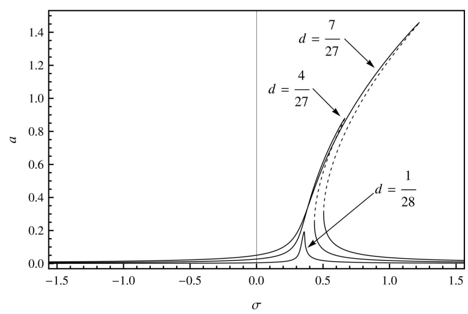

Fig. 7. The frequency response curves for different values of d

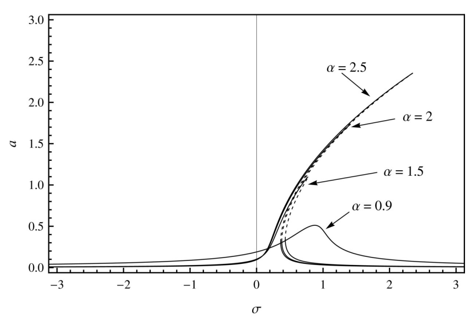

Fig. 8. The frequency response curves for different values of α

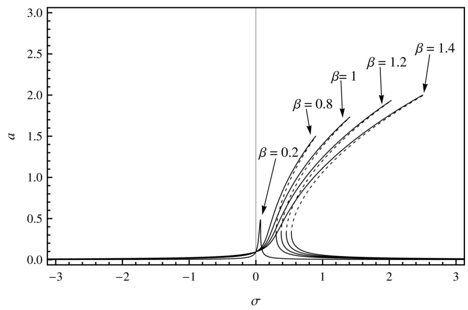

Fig. 9. The frequency response curves for different values of β

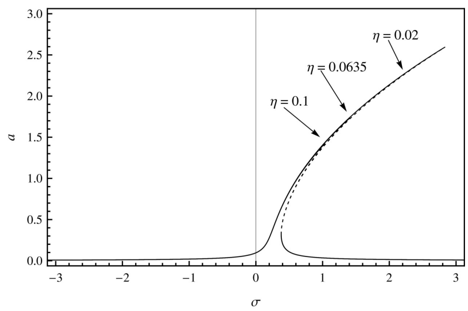

Fig. 10. The frequency response curves for different values of η

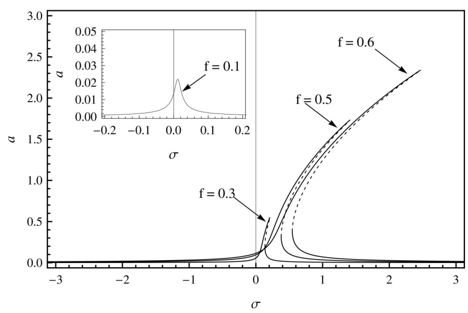

Fig. 11. The frequency response curves for different values of f

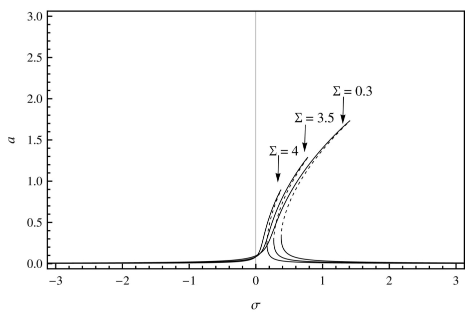

Fig. 12. The frequency response curves for different values of Σ

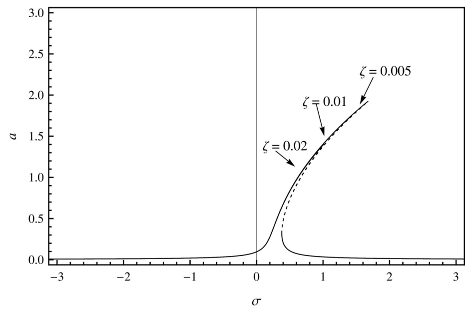

Fig. 13. The frequency response curves for different values of ζ

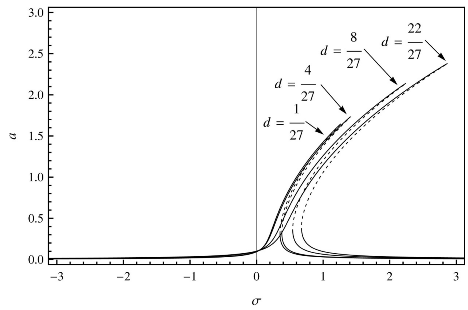

Fig. 14. The frequency response curves for different values of d

5 CONCLUSION

In this paper, we have investigated an analysis of superharmonic solutions of order 2 and 3 for a weakly nonlinear second order differential equation which represented the dynamic behavior of a microcantilever based on TM (Tapping mode) AFM (Atomic force microscopy). Two first order ordinary differential equations which describe the modulation of the amplitude and the phase are solved by using the method of multiple scales. Steady-state solution and its stability are investigated. Peak amplitude and its localization are determined. Numerical solutions of the frequency response equation and the stability equation are carried out for different values of the parameters in the equation. Results are represented in a group of figures in which solid curves (dashed) are stable (unstable) solutions.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

References

[1] Binnig G, Quate C. Atomic force microscope. Phys. Rev. Lett. 1986;56:9303.

[2] Rutzel S, Lee SI, Raman A. Nonlinear dynamics of atomic force microscope probes driven in Lennard-Jones potentials. Proc. R. Soc. Lond. A. 2003;459:1925-1948.

[3] Jalili N, Laxminarayana K. A review of atomic force microscopy imaging systems: Application to molecular metrology and biological sciences. International Journal of Mechanics. 2004;14(8):907945.

[4] Wagner R, Killgore JP, Tung RC, Raman A, Hurley DC. Vibrational shape tracking of atomic force microscopy cantilevers for improved sensitivity and accuracy of nano-mechanical measurements. Nanotechnology. 2015;26(045701):9.

[5] Berg J, Briggs GAD. Nonlinear dynamics of intermittent-contact mode atomic force microscopy. Physical Review B. 1997;55(22).

[6] Reithmeier E. Periodic solutions of nonlinear dynamical systems. Lecture Notes in Mathematics, Springer, New York; 1991.

[7] Nayfeh AH, Mook DT. Nonlinear oscillations. Wiley, New York; 1995.

[8] Nayfeh AH, Mook DT. Perturbation method. Willy Weinheim; 2004.

[9] Elnaggar AM, El-Bassiouny AF. Harmonic, subharmonic, superharmonic, simultaneous sub/superharmonic and combination responances of self-excited two coupled second order systems to multifrequency excitations. Acta Mechanica Sinica. 1993;8(1):6171.

[10] Piccirillo V, Balthazar JM, Jr BRP. Analytical study of the nonlinear behavior of a shape memory oscillator: Part IIresonance secondary. Nonlinear Dyn. 2010;60:513524.

[11] Shooshtari A, Pasha Zanoosi AA. A multiple times scale solution for non-linear vibration of mass grounded system. Applied Mathematical Modelling. 2010;34:19181929.

[12] Caruntu DI, Knechty MW. On nonlinear response near half natural frequency of electrostatically actuated micro resonators. International Journal of Structural Stability and Dynamics. 2011;11(4):641-672.

[13] Elnaggar AM, El-Bassiouny AF, Khalil KM. Saddle node bifurcation control for an odd nonlinearity problem. Global J. of Pure and Applied Mathematics. 2011;7(2):213-229.

[14] Ji JC, Zhang N. Suppression of superharmonic resonance response using a linear vibration absorber. Mechanics Research Communications. 2011;38:411416.

[15] Kim P, Bae S, Seok J. Resonant behaviors of a nonlinear cantilever beam with tip mass subject to an axial force and electrostatic excitation. International Journal of Mechanical Sciences. 2012;64:232-257.

[16] Elnaggar AM, Khalil KM. The response of nonlinear controlled system under an external excitation via time delay state feedback. Journal of King Saud University Engineering Sciences. 2016:28(1):75-83.

[17] Elnaggar AM, El-Bassiouny AF, Mosa GA. Perturbation analysis of an electrostatic Micro-Electro-Mechanical System (MEMS) subjected to external and non-linear parametric excitations. International Journal of Basic and Applied Sciences. 2014;3(3):209-223.

[18] Elnaggar AM, El-Bassiouny AF, Khalil KM, Omran AM. Periodic solutions of a modified Duffing equation subjected to a bi-harmonic parametric and external excitations. British Journal of Mathematics and Computer Science. 2016;16(4):1-12.

[19] Zhang WM, Meng G, Zhou JB, Ghen JY. Nonlinear dynamics and chaos of microcantilever-based TM-AFMs with squeeze film damping effects. Sensors. 2009;9:3854-3874.

[20] Elnaggar AM, Khalil KM, Rahby AS. Harmonic solution of a weakly nonlinear second order differential equation governed the motion of a TM-AFM cantilever. British Journal of Mathematics and Computer Science. 2016;15(4):1-11.

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