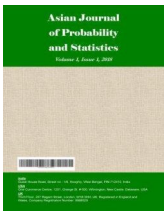

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On the Properties and Applications of Lomax-Exponential Distribution

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Authors' contributions

This work was carried out in collaboration between both authors. Author TGI designed the study, performed the statistical analysis, wrote the protocol, and wrote the first draft of the manuscript. Author DAK managed the analyses of the study and the literature searches. Both authors read and approved the final manuscript.

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Abstract

The Exponential distribution is memoryless and has a constant failure rate which makes it unsuitable for real life problems. This paper introduces a new distribution powered by an exponential random variable which gives a more flexible model for modelling real-life data. This new extension of the Exponential Distribution is called "Lomax-Exponential distribution (LED)". The extension of the new distribution became possible with the help of a Lomax generator proposed by [1]. This paper derives and studies some expressions for various statistical properties of the new distribution including distribution function, moments, quantile function, survival function and hazard function known as reliability functions. The inference for the Lomax-Exponentially distributed random variable were investigated based on some plots of the distribution and others revealed its behaviour and usefulness in real life situations. The parameters of the distribution are estimated using the method of maximum likelihood estimation. The performance of the new Lomax-Exponential distribution has been tested and compared to the Weibull-Exponential, Transmuted Exponential and the conventional Exponential distribution using three real life data sets.

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1. Introduction

The Exponential distribution has just a single parameter and it describes the time between events in a Poisson process. Apart from its usage in Poisson processes, it has been used extensively in the literature for life testing. The Exponential distribution is memoryless and has a constant failure rate; this latter property makes the distribution unsuitable for real life problems with bathtub failure rates [2] and inverted bathtub failure rates, hence there is a need to generalize the Exponential distribution in order to increase its flexibility and capability to model some other real life problems. There are several extensions of the exponential distribution. Some of the recent studies include the transmuted exponential distribution [3], transmuted inverse exponential distribution [4] and the Weibull-Exponential distribution by Oguntunde et al. [5].

The cumulative distribution function (*cdf*) and probability density function (*pdf*) of an exponential random variable *X* are respectively given by;

$$G(x) = 1 - \exp\{-\lambda x\} \tag{1.1}$$

$$g(x) = \lambda \exp\{-\lambda x\} \tag{1.2}$$

where $\lambda > 0$ is the exponential parameter and $x > 0$ is the random variable.

According to Cordeiro et al. [1] the *cdf* and *pdf* of the Lomax-G family (Lomax-based generator) for any continuous probability distribution are given respectively as:

$$F(x) = 1 - \beta^{\alpha}\left(\beta - \log\left[1 - G(x)\right]\right)^{-\alpha} \tag{1.3}$$

$$f(x) = \alpha\beta^{\alpha} g(x)\left(\left[1 - G(x)\right]\left\{\beta - \log\left[1 - G(x)\right]\right\}^{\alpha+1}\right)^{-1}, \tag{1.4}$$

where g(*x*) and G(*x*) are the *pdf* and *cdf* of any continuous distribution to be generalized respectively and α >0 and β>0 are the two additional new parameters responsible for the scale and shape of the distribution respectively.

Lomax [6] proposed an important probability distribution called Lomax distribution which has vast applications in lifetime and stochastic modelling of decreasing failure rate. Today, the distribution is widely used in studies of income, wealth inequality, and sizes of cities, queuing theory, and engineering, agricultural and biological analysis.

Studies conducted on Lomax distribution by several authors have been documented in the literature. Balakrishnan & Ahsanullah [7] discussed some important properties and moments of Lomax distribution. Al-Awadhi & Ghitany [8] provided the discrete Poisson-Lomax distribution. Abd-Elfattah et al. [9] studied the Bayesian and non-Bayesian estimation procedure of the reliability of Lomax distribution. Marshall-Olkin extended Lomax distribution that was introduced by Ghitany et al. [10]. The optimal times of changing stress level for simple stress plans under a cumulative exposure model for the Lomax distribution was determined by [11]. Hassan et al. [12] studied the optimal times of changing stress level for *k*-level step stress accelerated life tests based on adaptive type-II progressive hybrid censoring with product's lifetime following Lomax distribution.

Several authors have constructed various extensions of the Lomax distribution. For example, [13] proposed the exponentiated Lomax (*EL*) by introducing a new shape parameter to the existing Lomax distribution. Lemonte & Cordeiro [14] studied beta Lomax, Kumaraswamy Lomax and McDonald Lomax distributions. Cordeiro et al. [15] suggested the gamma-Lomax distribution. Rajab et al. [16] investigated a five-parameter beta Lomax distribution. Tahir et al. [17,18] introduced the Weibull Lomax distribution and the Gumbel-Lomax distribution. Recently [19] proposed the Exponentiated Lomax Geometric distribution.

In this paper, we present a new generalization of the exponential distribution called the Lomax-Exponential distribution (*LED*) using the proposed family by Cordeiro et al. [1]. The rest of the paper comprises the following sections: section 2 defines the *LED* with graphics for its *pdf* and *cdf*. Section 3 proposes some reliability functions of the new distribution. The maximum likelihood estimates (*MLEs*) of the unknown model parameters are obtained in section 4 while in section 5, we draw some applications of the *LED* using three real life datasets with concluding remarks in section 5.

2. Construction of Lomax-Exponential Distribution (LED)

Taking the *cdf* and the *pdf* of the Exponential distribution in equation (1.1) and (1.2) respectively. The *cdf* and *pdf* of the *LED* are obtained respectively from equation (1.3) and (1.4) as follows:

$$F(x)=1-\beta^{\alpha}\left(\beta-\log\left[1-G(x)\right]\right)^{-\alpha}$$

$$F(x)=1-\beta^{\alpha}\left(\beta-\log\left[1-\left(1-\exp\{-\lambda x\}\right)\right]\right)^{-\alpha}\quad F(x)=1-\beta^{\alpha}\left(\beta-\log\left[\exp\{-\lambda x\}\right]\right)^{-\alpha}$$

$$F(x)=1-\beta^{\alpha}\left(\beta+\lambda x\right)^{-\alpha}$$

$$F(x)=1-\beta^{\alpha}\left(\beta+\lambda x\right)^{-\alpha} \tag{2.1}$$

$$f(x)=\alpha\beta^{\alpha}g(x)\left(\left[1-G(x)\right]\left\{\beta-\log\left[1-G(x)\right]\right\}^{\alpha+1}\right)^{-1},$$

$$f(x)=\alpha\beta^{\alpha}\lambda\exp\{-\lambda x\}\left(\left[1-\left(1-\exp\{-\lambda x\}\right)\right]\left\{\beta-\log\left[1-\left(1-\exp\{-\lambda x\}\right)\right]\right\}^{\alpha+1}\right)^{-1},$$

$$f(x)=\alpha\beta^{\alpha}\lambda\exp\{-\lambda x\}\left(\exp\{-\lambda x\}\left\{\beta-\log\left[\exp\{-\lambda x\}\right]\right\}^{\alpha+1}\right)^{-1},$$

$$f(x)=\frac{\alpha\beta^{\alpha}\lambda\exp\{-\lambda x\}\left(\left\{\beta-\log\left[\exp\{-\lambda x\}\right]\right\}^{\alpha+1}\right)^{-1}}{\exp\{-\lambda x\}},$$

$$f(x)=\alpha\beta^{\alpha}\lambda\left(\beta-\log\left[\exp\{-\lambda x\}\right]\right)^{-\alpha-1},$$

$$f(x)=\alpha\beta^{\alpha}\lambda\left(\beta+\lambda x\right)^{-(\alpha+1)} \tag{2.2}$$

The following are the graphical representations of the *pdf* and *cdf* of the Lomax-Exponential distribution. Given some values of the parameters $a=\alpha, b=\beta \,\&\, l=\lambda$, we provide some possible graphs for the *pdf* and the *cdf* of the *LED* as shown in Figs. 1 and 2.

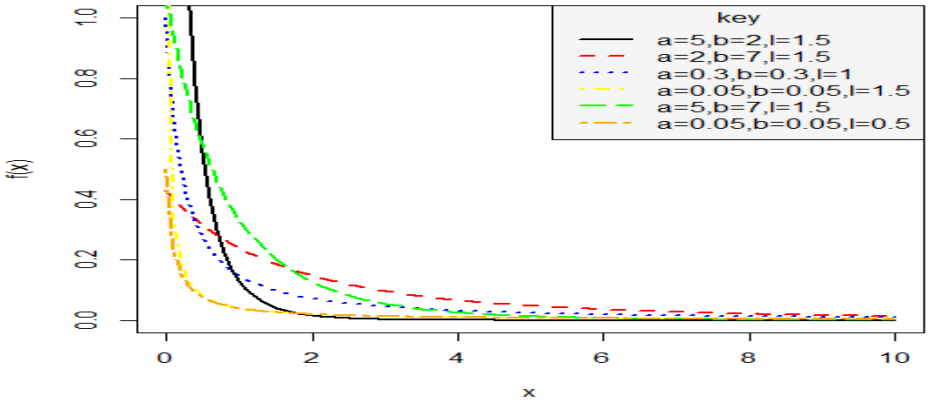

Fig. 1. Graph of *PDF* of the *LED* for different parameter values

Fig. 1 indicates that the *LED* is a skewed distribution and such skewed to the right. This means that distribution can be very useful for datasets that are positively skewed.

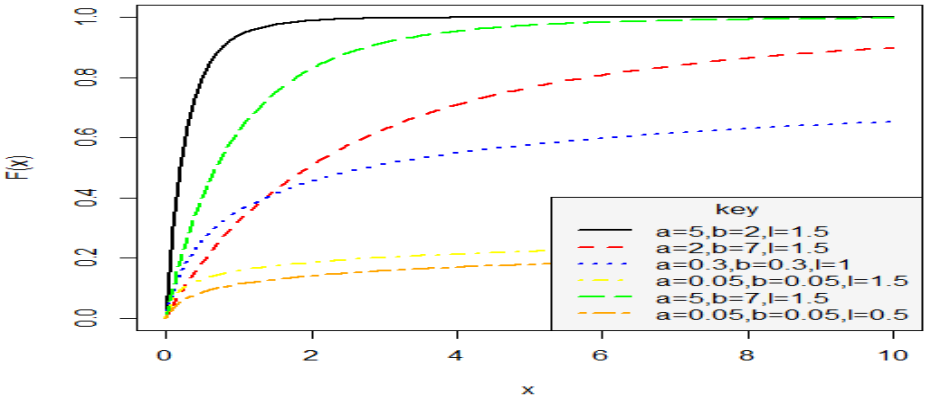

Fig. 2. Graph of *CDF* of the *LED* for different parameter values

From the Fig. 2 *cdf* plot, the *cdf* increases when *X* increases, and approaches 1 when *X* becomes large, as expected.

3. Statistical Properties of the Led

3.1 The Quantile Function

This function is derived by inverting the *cdf* of any given continuous probability distribution. It is used for obtaining some moments like skewness and kurtosis as well as the median and for generation of random variables from the distribution in question. Hyndman and Fan [20] defined the quantile function for any distribution in the form

$Q(u) = F^{-1}(u)$ where Q(u) is the quantile function of $F(x)$ for $0 < u < 1$

Taking *F(x)* to be the *cdf* of the Lomax-Exponential distribution and inverting it as above will give us the quantile function as follows:

$$F(x) = 1 - \beta^{\alpha}\left(\beta + \lambda x\right)^{-\alpha} = u \tag{3.1}$$

Simplifying equation (3.1) above, we obtain:

$$Q(u) = X_q = \lambda^{-1}\left[\left(\beta^{-\alpha}\left(1-u\right)\right)^{-\frac{1}{\alpha}} - \beta\right] \tag{3.2}$$

3.2 Skewness and Kurtosis

This paper presents the quantile-based measures of skewness and kurtosis due to the non-existence of the classical measures in some cases.

The Bowley's measure of skewness [21] based on quartiles is given by:

$$SK = \frac{Q\left(3/4\right) - 2Q\left(1/2\right) + Q\left(1/4\right)}{Q\left(3/4\right) - Q\left(1/4\right)} \tag{3.3}$$

And the [22] kurtosis is on octiles and is given by;

$$KT = \frac{Q\left(7/8\right) - Q\left(5/8\right) - Q\left(3/8\right) + \left(1/8\right)}{Q\left(6/8\right) - Q\left(1/8\right)} \tag{3.4}$$

3.3 Moments

Let X denote a continuous random variable, the n^{th} moment of X is given by;

$$\mu_n^{'} = E\left[X^n\right] = \int_0^{\infty} x^n f(x)dx \tag{3.5}$$

Taking *f(x)* to be the *pdf* of the *LED* as given in equation (2.1) and simplifying the integral we have:

$$\mu_n^{'} = E\left[X^n\right] = \int_0^{\infty} x^n\left(\alpha\beta^{\alpha}\lambda\left(\beta + \lambda x\right)^{-(\alpha+1)}\right)dx$$

$$\mu_n^{'} = E\left[X^n\right] = \alpha\beta^{\alpha}\lambda\int_0^{\infty} x^n\left(\beta + \lambda x\right)^{-(\alpha+1)}\,dx$$

Using integration by substitution, let:

$$u = \beta + \lambda x \Rightarrow x = -\tfrac{\beta}{\lambda}\left(1 - \tfrac{u}{\beta}\right)$$

$$\frac{du}{dx} = \lambda \Rightarrow dx = \frac{du}{\lambda}$$

Now, substituting for u, x and dx above, we have:

$$\mu_n^{'} = E\left[X^n\right] = \alpha\beta^{\alpha}\lambda\int_0^{\infty}\left(-\frac{\beta}{\lambda}\left(1-\frac{u}{\beta}\right)\right)^n (u)^{-(\alpha+1)}\frac{du}{\lambda}$$

$$\mu_n^{'} = E\left[X^n\right] = \frac{\alpha\beta^{\alpha+n}}{\lambda^n}(-1)^n\int_0^{\infty}(u)^{-(\alpha+1)}\left(1-\tfrac{u}{\beta}\right)^n du$$

$$\mu_n^{'} = E\left[X^n\right] = \frac{\alpha\beta^{n-1}}{\lambda^n}(-1)^n\int_0^{\infty}\left(\tfrac{u}{\beta}\right)^{1-\alpha-1-1}\left(1-\tfrac{u}{\beta}\right)^{n+1-1} du$$

Recall that $B(x,y) = B(y,x) = \int_0^{\infty} t^{x-1}(1-t)^{y-1}\,dt$ and this implies that

$$\mu_n^{'} = E\left(X^n\right) = \left(\frac{\alpha}{\beta}\right)\left(-\frac{\beta}{\lambda}\right)^n B(1-\alpha-1, n+1) \tag{3.6}$$

The mean, variance, skewness and kurtosis measures can also be calculated from the n^{th} ordinary moments as well as the moment generating function and characteristics function using some well-known relationships.

4. Some Reliability Functions

In this section, we present some reliability functions associated with *LED* including the survival and hazard functions.

4.1 The Survival Function

The survival function describes the likelihood that a system or an individual will not fail after a given time. It tells us about the probability of success or survival of a given product or component. Mathematically, the survival function is given by:

$$S(x) = 1 - F(x) \tag{4.1}$$

Taking *F(x)* to be the *cdf* of the Lomax-Exponential distribution, substituting and simplifying (4.1) above, we get the survival function of the *LED* as:

$$S(x) = 1-\left\{1-\beta^{\alpha}\left(\beta+\lambda x\right)^{-\alpha}\right\}$$

$$S(x) = \beta^{\alpha}\left\{\beta+\lambda x\right\}^{-\alpha} \tag{4.2}$$

Below is a plot of the survival function at chosen parameter values in Fig. 3.

From the Fig. 3, we observed that the probability of survival for any random variable following a Lomax-Exponential distribution drops as the time increases, that is, as time or age grows the probability of life or survival decreases. This implies that the Lomax-Exponential distribution could be used to model random variables whose survival rate decreases as their age or time grows.

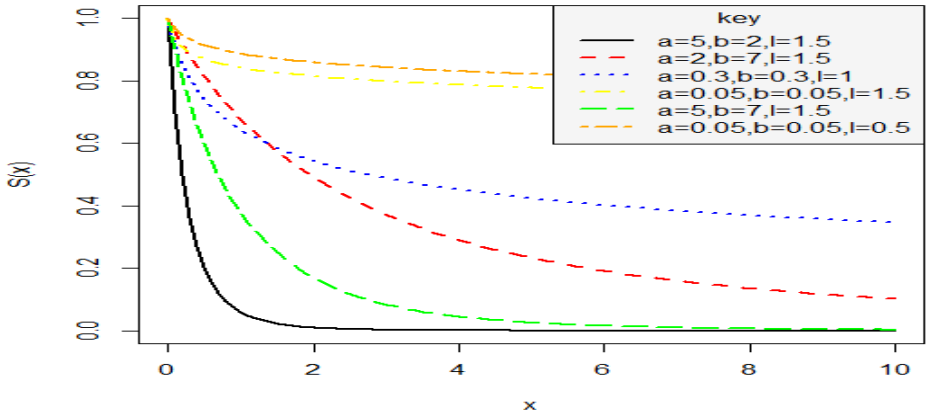

Fig. 3. Plot of the survival function of the *LED* for different parameter values

4.2 The Hazard Function

Hazard function as the name implies is also called risk function, it gives us the probability that a component will fail or die for an interval of time. The hazard function is defined mathematically as;

$$h(x)=\frac{f(x)}{1-F(x)}=\frac{f(x)}{S(x)} \tag{4.3}$$

Taking *f(x)* and *F(x)* to be the *pdf* and *cdf* of the proposed Lomax-Exponential distribution given previously, we obtain the hazard function as:

$$h(x)=\frac{\alpha\beta^{\alpha}\lambda\left(\beta+\lambda x\right)^{-(\alpha+1)}}{\beta^{\alpha}\left\{\beta+\lambda x\right\}^{-\alpha}}=\alpha\lambda\left\{\beta+\lambda x\right\}^{-\alpha-1-(-\alpha)}$$

$$h(x)=\alpha\lambda\left\{\beta+\lambda x\right\}^{-1} \tag{4.4}$$

The following is a plot of the hazard function at chosen parameter values in Fig. 4.

Fig. 4 shows the behaviour of hazard function of the LED and it means that the probability of failure for any random variable following a *LED* decreases as the values of the random variable increases, that is, as the time increases, the probability of failure or death decreases.

5. Parameter Estimation via Maximum Likelihood

Let $X_1,\ldots,X_n$ be a sample of size 'n' independently and identically distributed random variables from the *LED* with unknown parameters α, β, and λ defined previously. The *pdf* of the *LED* is given as:

$$f(x)=\alpha\beta^{\alpha}\lambda\left(\beta+\lambda x\right)^{-(\alpha+1)}$$

The likelihood function is given by;

$$L(X_1, X_2, ..., X_n / \alpha, \beta, \lambda) = \left(\alpha\beta^{\alpha}\lambda\right)^n \prod_{i=1}^{n} (\beta + \lambda x)^{-(\alpha+1)} \tag{5.1}$$

Taking the natural logarithm of the likelihood function, i.e.,

Let $l(n) = \log L(X_1, X_2, ..., X_n / \alpha, \beta, \lambda)$, such that

$$l = n\log\alpha + n\alpha\log\beta + n\log\lambda - (\alpha+1)\sum_{i=1}^{n}\log(\beta + \lambda x) \tag{5.2}$$

Differentiating $l(n)$ partially with respect to α, β *and* λ respectively gives;

$$\frac{\partial l(n)}{\partial \alpha} = \frac{n}{\alpha} + n\log\beta - \sum_{i=1}^{n}\log(\beta + \lambda x) \tag{5.3}$$

$$\frac{\partial l(n)}{\partial \beta} = \frac{n\alpha}{\beta} - (\alpha+1)\sum_{i=1}^{n}\left\{(\beta + \lambda x)^{-1}\right\} \tag{5.4}$$

$$\frac{\partial l(n)}{\partial \lambda} = \frac{n}{\lambda} - (\alpha+1)\sum_{i=1}^{n}\left\{x_i(\beta + \lambda x)^{-1}\right\} \tag{5.5}$$

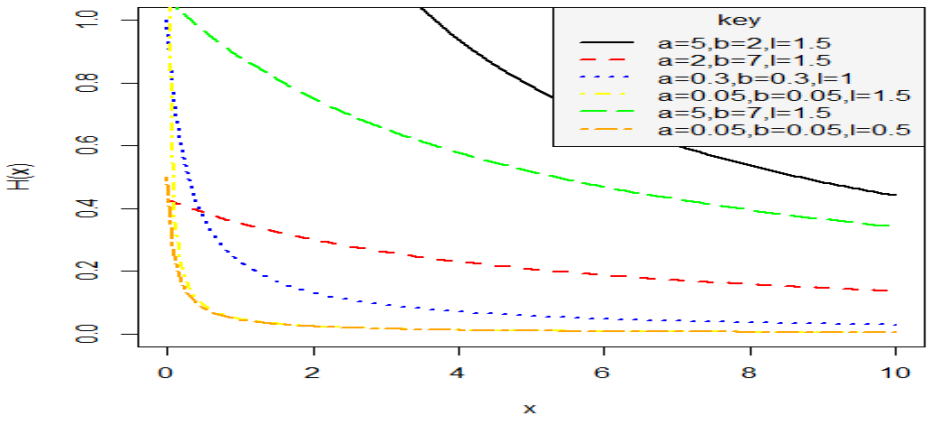

Fig. 4. Plot of the hazard function of the *LED* for different parameter values

Equating equations (5.3), (5.4) and (5.5) to zero and solving for the solution of the non-linear system of equations will give us the maximum likelihood estimates of parameters $\alpha, \beta \,\&\, \lambda$ respectively. However, the solution cannot be obtained analytically except numerically with the aid of suitable statistical software like Python, R, SAS, etc., when data sets are given.

6. Applications

This section examines the flexibility of *LED* while comparing its performance with other distributions such as the Weibull-exponential distribution (*WED*) and the exponential distribution (*ED*). Three real life data sets are employed to show how *LED* can be applied in practice with better performance than other distributions.

Data set I: This data set represents the waiting times (in minutes) before service of 100 Bank customers which was examined and analyzed by Ghitany et al. [23] for evaluating the performance of the Lindley distribution. It is as follows: 0.8,0.8,1.3,1.5,1.8,1.9,1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2, 3.3, 3.5, 3.6, 4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27, 31.6, 33.1, 38.5. The computed summary statistics of this dataset is given in Table 1.

Data Set II: This data set is the strength data of glass of the aircraft window reported by Fuller et al. [24]. This data is as follows: 18.83, 20.8, 21.657, 23.03, 23.23, 24.05, 24.321, 25.5, 25.52, 25.8, 26.69, 26.77, 26.78, 27.05, 27.67, 29.9, 31.11, 33.2, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.38. Its summary is computed and reported in Table 2.

Data set III: This data set represents the relief times (in minutes) of 20 patients receiving an analgesic reported by Gross and Clark [25] and has been used by Shanker et al. [26].

It is as follows: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0. The summary of the data set is provided in Table 3.

We also provide some histograms and densities for the three data sets as shown in Figs. 5, 6 and 7 respectively.

From the descriptive statistics in Tables 1, 2 and 3 and the histograms and densities shown in Figs. 5, 6 and 7 for the three datasets respectively, we observed that the three data sets are positively skewed, though, the third data set is highly peaked with a higher skewness coefficient followed by the first and then the second with a very low peak.

To evaluate the performance of these distributions, we have considered a goodness-of-fit test in order to know which distribution has a better fit given some datasets. Hence, we apply the Kolmogorov-Smirnov *(K-S)* test statistic. Further information about this statistic can be obtained from [27]. These statistics can be computed as:

$$K-S = D = \sup\left|F_n(x) - F_0(x)\right| \tag{6.1}$$

where $F_n(x)$ is the empirical distribution function and n is the sample size

Note: In decision making, a model with the lowest values for these statistics would be chosen as the best model to fit the data set in question.

Table 1. Summary statistics for data set I

Param.	N	Min	Q_1	Median	Q_3	Mean	Max.	Var.	Skew	Kurt.
Values	100	0.80	4.675	8.10	13.02	9.877	38.500	52.3741	1.4953	5.7345

Table 2. Summary Statistics for data set II

Param.	N	Min.	Q_1	Median	Q_3	Mean	Max.	Var.	Skew.	Kurt.
Values	31	18.83	25.51	29.90	35.83	30.81	45.38	52.61	0.43	2.38

Table 3. Summary Statistics for the Data set III

Param.	N	Min.	Q_1	Median	Q_3	Mean	Max.	Var.	Skew.	Kurt.
Values	20	1.10	1.475	1.70	2.05	1.90	4.10	0.4958	1.8625	7.1854

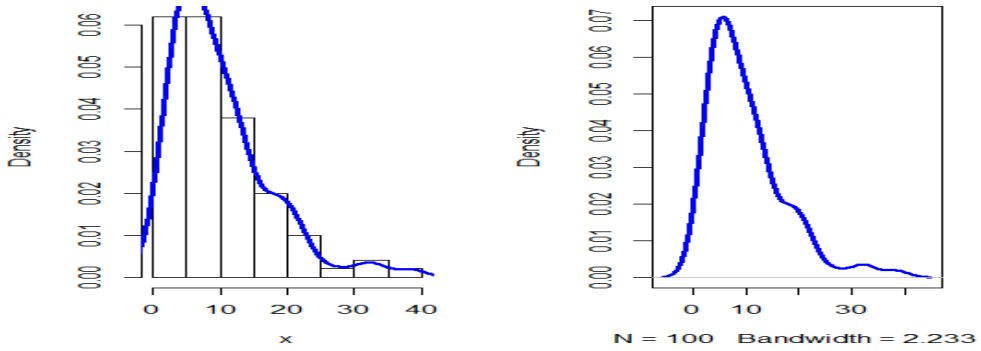

Fig. 5. A histogram and density plot for the waiting times of bank customers (Dataset I)

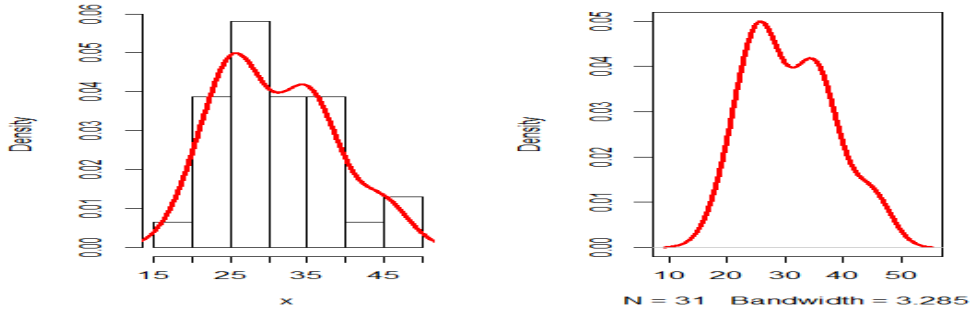

Fig. 6. A Histogram and density plot for the strength of glass of windows (Data set II)

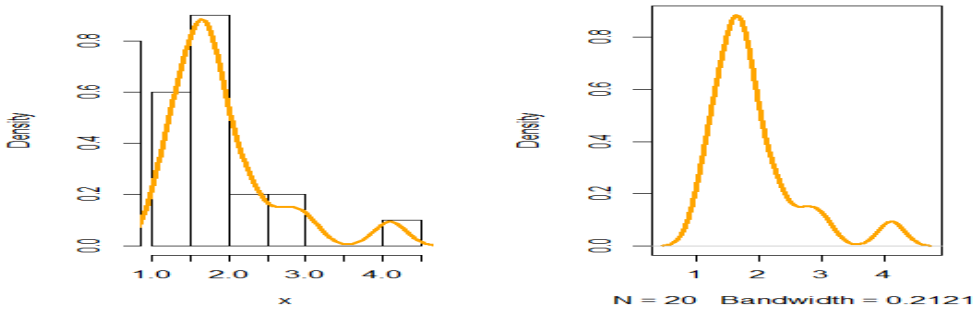

Fig. 7. A Histogram and density plot for the Relief times of 20 patients (Data set III)

The Table 4 shows the *K-S* values of the models with their corresponding p-values based on datasets I, II and III. From the result of Table 4, it is clearly seen that the *LED* has smaller or lower values of the *K-S* statistic with higher p-values for all the three datasets which is an indication that it has a better performance compared to the *WED* and the *ED*. Hence, we can confidently conclude that the *LED* is better than the *WED* and the *ED*. This also provides additional evidence to the fact that generalizing probability distributions provides compound distributions that are more flexible and better compared to their conventional counterparts.

Table 4. Performance evaluation of the *LED* with some generalizations of the Exponential distribution using the *K-S* values of the models with their corresponding p-values based on datasets I, II and III

Distributions	Dataset I	Dataset II	Dataset III	Ranks
LED	D = 0.1612 (0.01107)	D = 0.48213 (3.74*10^-07)	D = 0.43029 (0.001215)	1
WED	D = 0.55247 (2.2*10^-16)	D = 0.93961 (3.331*10^-16)	D = 0.34966 (0.01503)	2
ED	D = 0.9582 (2.2*10^-16)	D = 1.1842 (3.331*10^-16)	D = 0.62988 (2.564*10^-07)	3

Note: Values in parenthesis (.) are p-values

7. Conclusion

This paper presented a three-parameter Lomax-based Exponential distribution using a Lomax generator proposed by Cordeiro et al. [1]. Some statistical properties of the proposed distribution have been studied appropriately. We have derived explicit expressions for its quantile, moments, survival function and hazard functions with useful discussions. Some plots of the distribution indicated that the *LED* is a valid model and is skewed to the right. The implications of the plots for the survival and hazard functions showed that the *LED* could be appropriate for modelling time or age-dependent events, where survival and failure decrease with time or age. We estimated the model parameters using the method of maximum likelihood estimation. Finally, the performance of the new model has been tested based on some applications to three-lifetime datasets, and the results show that the *LED* performs satisfactorily better than the exponential distribution and Weibull-Exponential distribution.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Cordeiro GM, Ortega EMM, Popovic BV, Pescim RR. The Lomax generator of distributions: Properties, minification process and regression model. Appl. Maths. Comp. 2014;247:465-486.

[2] Singh SK, Singh U, Kumar M. Estimation of parameters of generalized inverted exponential distribution for progressive Type-II censored sample with binomial removals. J. Prob. Stat. 2013;183-652.

[3] Owoloko EA, oguntunde PE, Adejumo AO. Performance rating of the transmuted exponential distribution: an analytical approach. Spring. 2015;4:818-829.

[4] Oguntunde PE, Adejumo AO. The transmuted inverse exponential distribution. Inter. J. Adv. Stat. Prob. 2015;3(1):1–7.

[5] Oguntunde PE, Balogun OS, Okagbue HI, Bishop SA. The Weibull-exponential distribution: Its properties and applications. J. Appl. Sci. 2015;15(11):1305-1311.

[6] Lomax KS, Business failures: Another example of the analysis of failure data. J. Amer. Stat. Asso. 1954;49:847-852.

[7] Balakrishnan N, Ahsanullah M. Relations for single and product moments of record values from Lomax distribution. Sank. B. 1994;56:140-146.

[8] Al-Awadhi SA, Ghitany ME. Statistical properties of Poisson-Lomax distribution and its application to repeated accidents data. J. Appl. Stat. Sci. 2001;10(4):365-372.

[9] Abd-Elfattah AM, Alaboud FM, Alharby AH. On sample size estimation for Lomax distribution. Austr. J. Bas. Appl. Sci. 2007;1:373-378.

[10] Ghitany ME, Al-Awadhi FA, Alkhalfan LA. Marshall-Olkin extended Lomax distribution and its application to censored data. Comm. Stat.Theo. Meths. 2007;36:1855-1866.

[11] Hassan AS, Al-Ghamdi AS. Optimum step stress accelerated life testing for Lomax distribution. J. Appl. Sci. Res. 2009;5(12):2153-2164.

[12] Hassan AS, Assar SM, Shelbaia A. Optimum step stress accelerated life test plan for Lomax distribution with an adaptive type-II progressive hybrid censoring. Brit. J. Maths. Comp. Sci. 2016; 13(2):1-19.

[13] Abdul-Moniem IB, Abdel-Hameed HF. On exponentiated Lomax distribution. Inter. J. Math. Edu. 2012;33(5):1-7.

[14] Lemonte AJ, Cordeiro GM. An extended Lomax distribution. Statistics: A J. Theor. Appl. Stat. 2013;47:800-816.

[15] Cordeiro GM, Ortega EMM, Popović BV. The gamma-Lomax distribution. J. Stat. Comp. Simul. 2013;85(2):305–319.

[16] Rajab M, Aleem M, Nawaz T, Daniyal M. On five parameter beta Lomax distribution. J. Stat. 2013; 20:102-118.

[17] Tahir MH, Cordeiro GM, Mansoor M, Zubair M. The Weibull-Lomax distribution: Properties and applications. Hacett. J. Maths. Stat. 2015a;44(2):461-480.

[18] Tahir MH, Hussain MA, Cordeiro GM, Hamedani GG, Mansoor M, Zubair M. The Gumbel-Lomax distribution: Properties and applications. J. Stat. Theo. Appl. 2015b;15(1):61-79.

[19] Hassan AS, Abd-Allah M. Exponentiated Lomax Geometric Distribution: Properties and Applications. Pak. J. Stat. Oper. Res. 2017;XIII(3):545-566.

[20] Hyndman RJ, Fan Y. Sample quantiles in statistical packages, The Amer. Stat. 1996;50(4):361-365.

[21] Kenney JF, Keeping ES. Mathematics of statistics, 3rd edn, Chapman & Hall Ltd, New Jersey; 1962.

[22] Moors JJ. A quantile alternative for kurtosis. J. Roy. Stat. Soci.: Series D. 1998;37:25–32.

[23] Ghitany M, Al Mutairi D, Balakrishnan N, Al Enezi I. Power Lindley distribution and associated inference. Comp. Stat. Dat. Analy. 2013;64:20-33.

[24] Fuller EJ, Frieman S, Quinn J, Quinn G, Carter W. Fracture mechanics approach to the design of glass aircraft windows: A case study. SPIE Procs. 1994;2286:419-430.

[25] Gross AJ, Clark VA. Survival distributions reliability applications in the biometrical sciences. John Wiley, New York, USA; 1975.

[26] Shanker R, Fesshaye H, Sharma S. On two-parameter Lindley distribution and its applications to model lifetime data. Biomet. Biostat. Inter. J. 2016;3(1):39-56.

[27] Chen G, Balakrishnan N. A general purpose approximate goodness-of-fit test. J. Qual. Tech. 1995;27: 154-161.

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