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On Certain Properties on Vector - Valued Sequence Space on Product Normed Space

Jagat Krishna Pokhrel[1*]

[1]*Department of Mathematics, Sano Thimi Campus, Tribhuvan University, Kathmandu, Nepal.*

Author's contribution

This work was carried out after the studying some of the basic normed space valued sequence spaces and their linear topological structures. We characterize the linear space structures and containment relations on the space of sequences whose terms from a product normed space. These results can be used for further generalization to examine the properties of the various existing sequence spaces studied in Functional Analysis.

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Abstract

The notion of vector valued sequence space is a generalized form of spaces of scalar valued sequences, and its terms consist of sequences from a vector space. In this work, we shall study some conditions that characterize the linear space structures and containment relations of the space of sequences whose terms from a product normed space.

The aim of this paper is to deal with a vector valued sequence space $l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u})$ with its terms from a product normed space $A \times B$. We shall also investigate the linear space structure of $l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u})$ with respect to co-ordinatewise vector operations, the primary interest is to explore the conditions in terms of $\bar{u}$ and $\bar{\gamma}$ so that a class $l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u})$ is contained in or equal to another class of the same kind .

Keywords: Sequence space; generalized sequence space; product normed space.

**Corresponding author: E-mail: jagatpokhrel.tu@gmail.com;*

1 Introduction and Preliminaries

Let A be a normed space over $\mathbb{C}$, the field of complex numbers and let $\omega(A)$ denotes the linear space of all sequences $\overline{a} = (a_k)$ with $a_k \in A$, $k \geq 1$ with usual coordinatewise operations. We shall denote $\omega(\mathbb{C})$ by ω. Any subspace S of ω is then called a sequence space. A vector valued sequence space or a generalized sequence space is a linear space consisting of sequences with their terms from a vector space.

Let $(A, \|\,.\,\|_A)$ and $(B, \|\,.\,\|_B)$ be Banach spaces over the field $\mathbb{C}$ of complex numbers. Clearly the linear space structure of A and B provides the Cartesian product of A and B given by

$$A \times B = \{< a, b > : a \in A, b \in B\}$$

forms a normed linear space over $\mathbb{C}$ under the algebraic operations

$$< a_1, b_1 > + < a_2, b_2 > = < a_1 + a_2, b_1 + b_2 >$$
$$\text{and} \quad \alpha < a, b > = < \alpha a, \alpha b >$$

with the norm

$$\| < a, b > \| = \max \{\| a \|_A, \| b \|_B \},$$

where $< a_1, b_1 >, < a_2, b_2 >, < a, b > \in A \times B$ and $\alpha \in \mathbb{C}$.

Moreover since $(A, \|\,.\,\|_A)$ and $(B, \|\,.\,\|_B)$ are Banach spaces therefore $(A \times B, \| < .\,, . > \|)$ is also a Banach space.

The various types of vector valued sequence spaces has been significantly developed by several workers for instances, we refer a few [1,2,3,4,5,6,7,8].

Subsequently, in the works [9,10,11] and many others have introduced and examined some properties of bilinear vector valued sequence spaces defined on product normed space which generalize many sequence spaces.

2 The Vector Valued Sequence Space $l_\infty (A \times B, \|\,.\,\|, \overline{\gamma}, \overline{u})$

Let $\overline{u} = (u_k)$ and $\overline{v} = (v_k)$ be any sequences of strictly positive real numbers and $\overline{\gamma} = (\gamma_k)$ and $\overline{\mu} = (\mu_k)$ be sequences of non-zero complex numbers.

We now introduce and study the following class of Banach space $A \times B$ -valued sequences:

$$l_\infty (A \times B, \|\,.\,\|, \overline{\gamma}, \overline{u}) = \{\overline{u} = (< a_k , b_k >) : < a_k , b_k > \in A \times B, \quad \sup_k \| \gamma_k < a_k , b_k > \|^{u_k} < \infty\}.$$

Further, when $\gamma_k = 1$ for all k, then $l_\infty (A \times B, \|\,.\,\|, \overline{\gamma}, \overline{u})$ will be denoted by $l_\infty (A \times B, \|\,.\,\|, \overline{u})$ and when $u_k = 1$ for all k then $l_\infty (A \times B, \|\,.\,\|, \overline{\gamma}, \overline{u})$ will be denoted by $l_\infty (A \times B, \|\,.\,\|, \overline{\gamma})$.

3 Main Results

In this section we shall derive the linear space structure of the class $l_\infty (A \times B, \|\,.\,\|, \overline{\gamma}, \overline{u})$ over the field ***C*** of complex numbers and thereby investigate conditions in terms of $\overline{u}$, $\overline{v}$, $\overline{\gamma}$ and $\overline{m}$ so that a class is contained in or equal to another class of same kind. As far as the linear space structure of $l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u})$ over $\boldsymbol{C}$ is concerned we throughout take the co-ordinatewise vector operations i.e., for $\bar{w} = (< a_k, b_k >)$, $\bar{z} = (< a'_k, b'_k >)$ in $l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u})$ and scalar α, we have

$$\bar{w} + \bar{z} = (< a_k, b_k >) + (< a'_k, b'_k >) = (< a_k + a'_k, b_k + b'_k >)$$

and

$$\alpha\bar{u} = (\alpha < a_k, b_k >) = (< \alpha a_k, \alpha b_k >).$$

Further, by $\bar{u} = (u_k) \in \lambda_{\Box}$, we mean $\sup_k u_k < \infty$ and we see below that $\sup_k u_k < \infty$ is the necessary condition for linearity of the space.

We shall denote $M = \max (1, \sup_k u_k)$ and $A(\alpha) = \max(1, |\alpha|)$. The zero element of the space will be denoted by

$$\bar{\theta} = (< 0, 0 >, < 0, 0 >, < 0, 0 >, ...).$$

Theorem 1: $l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u})$ forms a linear space over C if and only if $\bar{u} = (u_k) \in \ell_\infty$.

Proof:

For the sufficiency, assume that $\bar{u} = (u_k) \in \lambda_\infty$ and $\bar{w} = (< a_k, b_k >)$ and $\bar{z} = (< a'_k, b'_k >) \in \lambda_\infty (A \times B, \bar{\gamma}, \bar{u}, \| . \|)$ .So that we have

$$\sup_k \| \gamma_k < a_k, b_k > \|^{u_k} < \infty \text{ and } \sup_k \| \gamma_k < a'_k, b'_k > \|^{u_k} < \infty.$$

Thus considering

$$\sup_k \| \gamma_k (< a_k, b_k > + < a'_k, b'_k >)\|^{u_k/M} \leq \sup_k \| \gamma_k < a_k, b_k > \|^{u_k/M} + \sup_k \| \gamma_k < a'_k, b'_k >\|^{u_k/M}$$

and we see that $\sup_k \| \gamma_k (< a_k, b_k > + < a'_k, b'_k >)\|^{u_k/M} < \infty$

and hence $\bar{w} + \bar{z} \in l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u})$. Similarly for any scalar α, we have $\alpha\bar{w} \in l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u})$ since

$$\begin{aligned} \sup_k \| \alpha\gamma_k < a_k, b_k >\|^{u_k/M} &= \sup_k |\alpha|^{u_k/M} \| \gamma_k < a_k, b_k > \|^{u_k/M} \\ &\leq A(\alpha) \sup_k \| \gamma_k < a_k, b_k > \|^{u_k/M} < \infty. \end{aligned}$$

Conversely if $\bar{u} = (u_k) \notin \ell_\infty$ then we can find a sequence $(k(n))$ of positive integers with

$$k(n) < k(n + 1), n \geq 1$$

such that $u_{k(n)} > n$ for each $n \geq 1$. Now taking $< r, t > \in A \times B$, $\| < r, t > \| = 1$ we define a sequence $\overline{w} = (< a_k , b_k >)$ by

$$< a_k , b_k > = \begin{cases} \lambda_{k(n)}^{-1} n^{-r_{k(n)}} < r, \; t >, \text{ for } k = k(n), n \geq 1 \;, \text{ and} \\ < 0, 0 >, \text{ otherwise.} \end{cases}$$

where $< r, t > \in A \times B$ with $\| < r, t > \| = 1$, then we have

$$\begin{aligned} \sup_k \| \gamma_k < a_k , b_k > \|^{u_k} &= \sup_n \| \gamma_{k(n)} < a_{k(n)}, b_{k(n)} > \|^{u_{k(n)}} \\ &= \sup_n \| n^{-r_{k(n)}} < r, t > \|^{u_{k(n)}} \\ &= \sup_n \frac{1}{n} = 1. \end{aligned}$$

Thus we easily see that $\overline{w} \in l_\infty (A \times B, \| \, . \, \| , \; \overline{\gamma}, \overline{u})$ but on the other hand for $k = k(n)$, $n \geq 1$ and for the scalar

$\alpha = 2$, we have

$$\begin{aligned} \sup_k \| \gamma_k (\alpha < a_k , b_k >) \|^{u_k} &= \sup_k \| \gamma_{k(n)} (\alpha < a_{k(n)} , b_{k(n)} >)\|^{u_{k(n)}} \\ &= \sup_n |2|^{u_{k(n)}} \; \| n^{-r_{k(n)}} < r, t > \|^{u_{k(n)}} \\ &= \sup_n |2|^{u_{k(n)}} \; . \frac{1}{n} > \sup_n \frac{2^n}{n} \geq 1 \end{aligned}$$

This shows that $\alpha \, \overline{w} \notin \ell_\infty (A \times B, \overline{\gamma}, \overline{u}, \| \, . \, \|)$. Hence $l_\infty (A \times B, \| \, . \, \| , \; \overline{\gamma}, \overline{u})$ will form linear space

if and only if $\overline{u} = (u_k) \in \ell_\infty$.

Theorem 2: For any $\overline{u} = (u_k)$, $l_\infty (A \times B, \| \, . \, \|, \; \overline{\gamma}, \overline{u}) \subset l_\infty (A \times B, \| \, . \, \|, \overline{\mu}, \overline{u})$ if and only if

$$\lim \sup_k \left| \frac{\gamma_k}{\mu_k} \right|^{u_k} > 0.$$

Proof :

Suppose $\lim \inf_k \left| \frac{\gamma_k}{\mu_k} \right|^{u_k} > 0$, and $\overline{w} = (< a_k , b_k >) \in l_\infty (A \times B, \| \, . \, \| , \; \overline{\gamma}, \overline{u})$. Then there exists $m > 0$, such that

$$m|\mu_k|^{u_k} < |\gamma_k|^{u_k}$$

for all sufficiently large values of k. Thus

$$\sup_k \| \mu_k < a_k , b_k > \|^{u_k} \leq \sup_k \frac{1}{m} \| \gamma_k < a_k , b_k > \|^{u_k} < \infty$$

for all sufficiently large values of k, implies that $\overline{w} \in l_\infty (A \times B, \| \, . \, \| , \; \overline{\mu}, \overline{u})$. Hence

$$l_\infty(A \times B, \|\,.\,\|, \bar{\gamma}, \bar{u}) \subset l_\infty(A \times B, \|\,.\,\|, \bar{\mu}, \bar{u}).$$

Conversely, let

$$l_\infty(A \times B, \|\,.\,\|, \bar{\gamma}, \bar{u}) \subset l_\infty(A \times B, \|\,.\,\|, \bar{\mu}, \bar{u})$$

but $\lim \inf_k \left|\frac{\gamma_k}{\mu_k}\right|^{u_k} = 0$. Then we can find a sequence $(k(n))$ of positive integers with

$$k(n) < k(n+1),\ n \geq 1$$

such that

$$|\mu_{k(n)}|^{u_{k(n)}} > n|\gamma_{k(n)}|^{u_{k(n)}}$$

So, if we take the sequence $\bar{w} = (< a_k, b_k >)$ defined by

$$< a_k, b_k > = \begin{cases} \gamma_{k(n)}^{-1} < r,\ t >, \text{ for } k = k(n),\ n \geq 1, \text{ and} \\ < 0,\ 0 >, \text{ otherwise.} \end{cases}$$

where $< r, t > \in A \times B$ with $\|< r, t >\| = 1$, then we easily see that

$$\sup_k \|\gamma_k < a_k, b_k >\|^{u_k} = \sup_n \|\gamma_{k(n)} < a_{k(n)}, b_{k(n)} >\|^{u_{k(n)}}$$
$$= \sup_n \|< r, t >\|^{u_{k(n)}} = 1$$

and ,

$$\sup_k \|\mu_k < a_k, b_k >\|^{u_k} = \sup_n \|\mu_{k(n)} < a_{k(n)}, b_{k(n)} >\|^{u_{k(n)}}$$
$$= \sup_n \left\{ \left|\frac{\mu_{k(n)}}{\gamma_{(n)}}\right|^{u_{k(n)}} \|< r, t >\|^{u_{k(n)}} \right\}$$
$$> \sup_n n = \infty.$$

Hence $\bar{w} \in l_\infty(A \times B, \|\,.\,\|, \bar{\gamma}, \bar{u})$ but $\bar{w} \notin l_\infty(A \times B, \|\,.\,\|, \bar{\mu}, \bar{u})$, a contradiction. This completes the proof.

Theorem 3: For any $\bar{u} = (u_k)$, $l_\infty(A \times B, \|\,.\,\|, \bar{\mu}, \bar{u}) \subset l_\infty(A \times B, \|\,.\,\|, \bar{\gamma}, \bar{u})$

if and only if $\lim \sup_k \left|\frac{\gamma_k}{\mu_k}\right|^{u_k} < \infty$.

Proof:

For the sufficiency, suppose $\lim \sup_k \left|\frac{\gamma_k}{\mu_k}\right|^{u_k} < \infty$, and $\bar{w} = (< a_k, b_k >) \in l_\infty(A \times B, \|\,.\,\|, \bar{\mu}, \bar{u})$.

Then there exists $L > 0$, such that

$$L|\mu_k|^{u_k} > |\gamma_k|^{u_k}$$

for all sufficiently large values of k. Thus

$$\sup_k \| \gamma_k < a_k , b_k > \|^{u_k} \leq \sup_k L \| \mu_k < a_k , b_k > \|^{u_k} < \infty,$$

for all sufficiently large values of k, implies that $\overline{w} \in l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$. Hence

$$l_\infty (A \times B, \| . \|, \overline{\mu}, \overline{u}) \subset l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u}).$$

For the necessity, suppose that

$$l_\infty (A \times B, \| . \| , \overline{\mu}, \overline{u}) \subset l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$$

but $\lim \sup_k \left| \frac{\gamma_k}{\mu_k} \right|^{u_k} = \infty$. Then we can find a sequence $(k(n))$ of positive integers

$$k(n) < k(n + 1), n \geq 1$$

such that

$$n|\mu_{k(n)}|^{u_{k(n)}} < | \gamma_{k(n)}|^{u_{k(n)}} , \text{ for each } n \geq 1$$

For $< r, t > \in A \times B$ with $\| < r, t > \| = 1$ we define sequence $\overline{w} = (< a_k , b_k >)$ such that

$$< a_k , b_k > = \begin{cases} \mu_{k(n)}^{-1} < r, t >, \text{ for } k = k(n), n \geq 1 , \text{ and} \\ < 0, 0 >, \text{ otherwise.} \end{cases}$$

Then we easily see that

$$\sup_k \| \mu_k < a_k , b_k > \|^{u_k} = \sup_n \| \mu_{k(n)} < a_{k(n)}, b_{k(n)} > \|^{u_{k(n)}} = \sup_n \| < r, t > \|^{u_{k(n)}} = 1$$

and

$$\sup_k \|\gamma_k < a_k , b_k > \|^{u_k} = \sup_n \|\gamma_{k(n)} < a_{k(n)}, b_{k(n)} >\|^{u_{k(n)}} = \sup_n \left\{ \left| \frac{\gamma_{k(n)}}{\mu_{(n)}} \right|^{u_{k(n)}} \| < r, t > \|^{u_{k(n)}} \right\} > \sup_n n = \infty.$$

Hence $\overline{w} \in l_\infty (A \times B, \| . \| , \overline{\mu}, \overline{u})$ but $\overline{w} \notin l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$, which leads to a contradiction.

This completes the proof.

When Theorems 2 and 3 are combined, we get

Theorem 4: For any $\overline{u} = (u_k)$, $l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u}) = l_\infty (A \times B, \| . \| , \overline{\mu}, \overline{u})$

if and only if $0 < \lim \inf_k \left| \frac{\gamma_k}{\mu_k} \right|^{u_k} \leq \lim \sup_k \left| \frac{\gamma_k}{\mu_k} \right|^{u_k} < \infty$.

Corollary: For any $\bar{u} = (u_k)$,

(i) $l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u}) \subset l_\infty(A \times B, \|.\|, \bar{u})$ if and only if $\lim \inf_k |\gamma_k|^{u_k} > 0$;

(ii) $l_\infty(A \times B, \|.\|, \bar{u}) \subset l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u})$ if and only if $\lim \sup_k |\gamma_k|^{u_k} < \infty$;

(iii) $l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u}) = l_\infty(A \times B, \|.\|, \bar{u})$ if and only if

$$0 < \lim \inf_k |\gamma_k|^{u_k} \le \lim \sup_k |\gamma_k|^{u_k} < \infty.$$

Proof:

Proof follows if we take $\mu_k = 1$ for all k in Theorems 2, 3 and 4.

Theorem 5: For any $\bar{\gamma} = (\gamma_k)$, $l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u}) \subset l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{v})$

if and only if $\lim \sup_k \frac{v_k}{u_k} < \infty$.

Proof:

Let the condition hold. Then there exists $L > 0$ such that $v_k < Lu_k$ for all sufficiently large values of k. Thus

$$\sup_k \| \gamma_k < a_k, b_k > \|^{u_k} \le N \text{ for some } N > 1$$

implies that

$$\sup_k \| \gamma_k < a_k, b_k > \|^{v_k} \le N^L,$$

and hence $l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{u}) \subset l_\infty(A \times B, \|.\|, \bar{\gamma}, \bar{v})$.

Conversely, let the inclusion hold but $\lim \sup_k \frac{v_k}{u_k} = \infty$. Then there exists a sequence $(k(n))$ of positive integers with

$$k(n) < k(n+1),\ n \ge 1$$

such that

$$v_{k(n)} > n\, u_{k(n)},\ n \ge 1.$$

We now define a sequence $\bar{w} = (< a_k, b_k >)$ as follows:

$$< a_k, b_k > = \begin{cases} \gamma_{k(n)}^{-1} 2^{1/u_{k(n)}} < r,\ t >, \text{ for } k = k(n),\ n \ge 1 \\ < 0,\ 0 >, \text{ otherwise.} \end{cases} \text{, and}$$

where $< r, t > \in A \times B$ with $\| < r, t > \| = 1$.

Then for $k = k(n)$, $n \ge 1$, we easily see that

$$\sup_k \| \gamma_k < a_k , b_k > \|^{u_k} = \sup_n \| \gamma_{k(n)} < a_{k(n)}, b_{k(n)} > \|^{u_{k(n)}}$$
$$= 2 \sup_n \| < r, t > \|^{u_{k(n)}} = 2$$

and , $$\sup_k \|\gamma_k < a_k , b_k > \|^{v_k} = \sup_n \|\gamma_{k(n)} < a_{k(n)}, b_{k(n)} >\|^{v_{k(n)}}$$
$$= \sup_n \| 2^{1/u_{k(n)}} < r, t > \|^{v_{k(n)}}$$
$$> \sup_n 2^n = \infty.$$

Hence $\overline{w} \in l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$ but $\overline{w} \notin l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{v})$, a contradiction.

This completes the proof.

Theorem 6: For any $\overline{\gamma} = (\gamma_k)$, $l_\infty (A \times B, \| . \|, \overline{\gamma}, \overline{v}) \subset l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$

if and only if $\lim \inf_k \frac{v_k}{u_k} > 0$.

Proof:

Let the condition hold and $\overline{w} = (< a_k , b_k >) \in l_\infty (A \times B, \| . \|, \overline{\gamma}, \overline{v})$ .Then there exists $m > 0$ such that $v_k < m\, u_k$ for all sufficiently large values of k and

$$\sup_k \| \gamma_k < a_k , b_k >\|^{v_k} \le N \text{ for some } N > 1.$$

This implies that

$$\sup_k \| \gamma_k < a_k , b_k >\|^{u_k} \le N^{1/m}$$

i.e, $\overline{w} = (< a_k , b_k >) \notin l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$ and hence

$$l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{v}) \subset l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u}).$$

Conversely let the inclusion hold but $\lim \inf_k \frac{v_k}{u_k} = 0$. Then we can find a sequence $(k(n))$ of positive integerswith $k(n) < k(n + 1), n \ge 1$

such that

$$n\, v_{k(n)} < u_{k(n)}, n \ge 1.$$

Now taking $< r, t > \in A \times B$ with $\| < r, t > \| = 1$, we define the sequence $\overline{w} = (< a_k , b_k >)$ by

$$< a_k , b_k > = \begin{cases} \gamma_{k(n)}^{-1} 2^{1/v_{k(n)}} < r, t >, \text{ for } k = k(n), n \ge 1 \text{ , and} \\ < 0, 0 >, \text{ otherwise.} \end{cases}$$

Then for $k = k(n)$, $n \geq 1$, we easily see that

$$\sup_k \| \gamma_k < a_k , b_k > \|^{v_k} = \sup_n \| \gamma_{k(n)} < a_{k(n)}, b_{k(n)} > \|^{v_{k(n)}}$$
$$= 2 \sup_n \| < r, t > \|^{v_{k(n)}} = 2$$

and $$\sup_k \|\gamma_k < a_k , b_k > \|^{u_k} = \sup_n \|\gamma_{k(n)} < a_{k(n)}, b_{k(n)} > \|^{u_{k(n)}}$$
$$= \sup_n \| 2^{1/v_{k(n)}} < r, t > \|^{u_{k(n)}}$$
$$> \sup_n 2^n = \infty.$$

Hence $\bar{w} \in l_\infty (A \times B, \| . \| , \bar{\gamma}, \bar{v})$ but $\bar{w} \notin l_\infty (A \times B, \| . \| , \bar{\gamma}, \bar{u})$, a contradiction.

This completes the proof.

On combining Theorems 5 and 6, we get the following theorem:

Theorem 7: For any $\bar{\gamma} = (\gamma_k)$, $l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u}) = l_\infty (A \times B, \| . \| , \bar{\gamma}, \bar{v})$ if and only if $0 < \lim \inf_k \frac{v_k}{u_k} \leq \lim \sup_k \frac{v_k}{u_k} < \infty$.

Corollary: For any $\bar{\gamma} = (\gamma_k)$,

(i) $l_\infty (A \times B, \| . \|, \bar{\gamma}) \subset l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u})$ if and only if $\lim \sup_k u_k < \infty$;

(ii) $l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u}) \subset l_\infty (A \times B, \| . \|, \bar{\gamma})$ if and only if $\lim \inf_k u_k > 0$;

(iii) $l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u}) = l_\infty (A \times B, \| . \|, \bar{\gamma})$ if and only if $0 < \lim \inf_k u_k \leq \lim \sup_k u_k < \infty$.

Proof:

Proof easily follows when we take $u_k = 1$ and $v_k = u_k$ for all k in theorem 5, 6 and 7.

Theorem 8: For any sequences $\bar{\gamma} = (\gamma_k)$, $\bar{\mu} = (\mu_k)$, $\bar{u} = (u_k)$ and $\bar{v} = (v_k)$,

$$l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u}) \subset l_\infty (A \times B, \| . \| , \bar{\mu}, \bar{v})$$

if and only if (i) $\lim \inf_k \left| \frac{\gamma_k}{\mu_k} \right|^{u_k} > 0$, and (ii) $\lim \sup_k \frac{v_k}{u_k} < \infty$.

Proof:

Proof directly follows from Theorems 2 and 5.

In the following example we show that $l_\infty (A \times B, \| . \|, \bar{\gamma}, \bar{u})$ is strictly contained in $l_\infty (A \times B, \| . \| , \bar{\gamma}, \bar{v})$ however (i) and (ii) of Theorem 8 are satisfied.

Example:

Let $\overline{w} = (< a_k , b_k >)$ be a sequence in Banach space $A \times B$ such that

$$\|< a_k , b_k >\| = k^k.$$

Take $u_k = k^{-1}$ if k is odd integer and $u_k = k^{-2}$, if k is even integer, $v_k = k^{-2}$ for all values of k, $\gamma_k = 3^k$ for all values of k; and $\mu_k = 2^k$, for all values of k. Then

$\left|\frac{\gamma_k}{\mu_k}\right|^{u_k} = \frac{3}{2}$ if k is odd integer and $\left|\frac{\gamma_k}{\mu_k}\right|^{u_k} = \left(\frac{3}{2}\right)^{1/2}$, if k is even integer.

Thus $\lim \inf_k \left|\frac{\gamma_k}{\mu_k}\right|^{u_k} = 1$ i.e. condition (i) of Theorem 8 is satisfied.

Further since $\frac{v_k}{u_k} = \frac{1}{k}$, if k is odd integer and $\frac{v_k}{u_k} = 1$, if k is even integer, therefore condition (ii) of Theorem 8 is also satisfied as $\lim \sup_k \frac{v_k}{u_k} = 1$.

We now see that $\overline{w} = (< a_k , b_k >) \in l_\infty (A \times B, \| . \|, \overline{\mu}, \overline{v})$ for all $k \geq 1$ as

$$\sup_k \| \mu_k < a_k , b_k > \|^{v_k} = \sup_k (2k)^{1/k} < 2,$$

but $\overline{w} = (< a_k , b_k >) \notin l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$, when k is odd integer as

$$\sup_k \| \gamma_k < a_k , b_k >\|^{u_k} = \sup_k 3k = \infty$$

This shows that the condition (i) and (ii) are satisfied but $l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{u})$ is strictly contained in

$$l_\infty (A \times B, \| . \| , \overline{\gamma}, \overline{v}).$$

4 Conclusion

This paper establishes some of the results that characterize the linear space structures and containment relations on the space of sequences whose terms from a product normed space. In fact, these results can be used for further generalization and unification to investigate the properties of the various existing sequence spaces studied in Functional Analysis.

Competing Interests

Author has declared that no competing interests exist.

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