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Special B-spline Tight Framelet and It's Applications

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

We present a special B-spline tight frame and use it to introduce our numerical approximation method. We apply our method to investigate Gibbs effects and illustrate some features of the associated framelet expansion. It is shown that Gibbs effects occurs in the framelet expansion of a function with a jump discontinuity at 0 for certain classes of framelets. Numerical results are obtained regarding the behavior of the Gibbs effects. We present the results by expanding functions using the quasi-affine system. This system is generated by the B-spline tight framelets with a specific number of generators. We show numerically the existence of Gibbs effects in the truncated expansion of a given function by using some tight framelet representation.

Keywords: *Gibbs phenomenon; Tight frames; Unitary extension principle (UEP); Quasi-affine system; B-splines.*

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1 Introduction

The theory and applications of approximating functions can be traced back to more than one hundred years ago [1]. Various methods and techniques have been developed in numerical approximations of functions. Here, we introduce a new method of approximating functions, by means of, framelet expansion. Frame is a generalization of a basis of a vector space or function space. It maybe linearly dependent and provides a redundant, stable way of representing a given function. For many decades, Fourier series was used to approximate functions in terms of simple oscillating sines and cosines. However, the nonuniform convergence of the classical Fourier series for functions with jump discontinuities was already analyzed by Wilbraham in 1848. This was later named the Gibbs phenomenon (effects). The partial sums overshoot the function near the discontinuity, and this overshoot continues no matter how many terms are taken in the partial sum. This phenomenon is not the special quirk which occurs only in trigonometric Fourier series. A similar phenomenon exists in other wavelet series [2, 3], and tight framelet expansions [4, 5]. The numerical approximation methodology is a broad subject. Here we mainly discuss its applications on the Gibbs phenomenon which opens a new insight of this study.

Gibbs effect is not simply a mathematical observation; it has many practical applications in applied science, computer and engineering. For example, The development of radar in Great Britain, engineers used the square wave function to give x-coordinates on their oscilloscopes regularly. In doing that, they encountered Gibbs effects [6]. In the digital signals [7], Gibbs effects appear in the process of the design of digital filters by electrical engineers.

Gibbs effects can be used as techniques for numerical analysis and computation [8, 9], vibration and stability of complex beams [10], pseudo-spectral time domain analysis [11], string approximation for spam filtering and malicious URLs classification system [12, 13], and cubic-spline interpolation of discontinuous functions [14]. It can be viewed as a phenomenon of recovering local information from global information, or, specifically, as one of recovering point values of a function from its expansion coefficients. In physical optics, for example, Gibbs effects displayed when a beam irradiance of a top-hat beam in an aperture is represented as a plot of the beam irradiance cross sectional surface past an aperture [15].

This article is organized as follows. In Section 2, we recall some preliminary backgrounds by introducing some notations. Section 3 provides some fundamentals in frame theory. We then begin the study of Gibbs effects by showing Bessel and Legendre series expansions in Section 4. The main ingredient in constructing wavelets is multiresolution analysis which is explained in Section 5. We present the unitary extension principle for constructing tight frames in Section 6. We also introduce framelet representation and use it to show how to obtain the truncated partial sum of a function. In Section 7, we present a quasi-affine system to construct a tight framelet. We further construct B-spline tight frame in Section 8. Moreover, we present the Gibbs effects in B-spline tight frames using three generators in Section 9. We conclude with several remarks in Section 10.

2 Preliminary Background

The space $L^2(\mathbb{R})$ is the set of all functions $f(x)$ such that

$$\|f\|_{L^2(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^2 \right)^{1/2} < \infty.$$

The main feature of a basis $\{f_k\}_{k=1}^{\infty}$ in the Hilbert space $L^2(\mathbb{R})$ is that every $f \in L^2(\mathbb{R})$ can be represented as an infinite linear combination of the elements f_k such that

$$f = \sum_{k=1}^{\infty} c_k f_k. \tag{2.1}$$

Let $\ell_2(\mathbb{Z})$ be the set of all sequences $h[k]$ defined on $\mathbb{Z}$ where:

$$\left(\sum_{k=-\infty}^{\infty} |h[k]|^2 \right)^{1/2} < \infty.$$

For any function $f \in L^2(\mathbb{R})$, the dyadic dilation operator D is defined by

$$Df(x) = \sqrt{2} f(2x).$$

For $a \in \mathbb{R}$, the operator T_a, called translation by a, is defined by

$$T_a f\,(x) = f(x-a),\ x \in \mathbb{R}.$$

Let $j \in \mathbb{Z}$, then we have

$$T_a D^j = D^j T_{2^j a} \text{ and } D^j T_a = T_{2^{-j} a} D^j,\ \text{ where } D^j = \underbrace{D \cdot D \cdots D}_{j\text{-}times}$$

3 Frames in $L^2(\mathbb{R})$

In this section we review some elements and basic information for the frame theory, which is an active research area dealing with a generalization of the concept of an orthonormal basis. The idea comes by having an additional lower bound of the Bessel sequences. Frames were introduced already in 1952 by Duffin and Schaeffer in their paper [16]; they used frames as a tool in the study of nonharmonic Fourier series. In 1985, Daubechies, Grossmann and Meyer [17] observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$.

Definition 3.1. *A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $L^2(\mathbb{R})$ is a frame for $L^2(\mathbb{R})$ if there exist constants $A, B > 0$ such that*

$$A\|f\|^2 \leq \sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2,\ \forall f \in L^2(\mathbb{R}). \tag{3.1}$$

The numbers A, B are called frame bounds.

A frame is *tight* if we can choose $A = B$ as a frame bounds, and it is *Parseval frame* if $A = B = 1$. All frames can be viewed as a generalization of an orthonormal basis. A frame, which is *not* a basis, is said to be *overcomplete* or *redundant.*

Definition 3.2. *A sequence $\{f_k\}_{k=1}^{\infty}$ of elements in $L^2(\mathbb{R})$ is a Bessel sequence if there exists a constant $B > 0$ such that*

$$\sum_{k=1}^{\infty} |\langle f, f_k \rangle|^2 \leq B\|f\|^2, \forall f \in L^2(\mathbb{R}),$$

and B is called the Bessel bound of $\{f_k\}_{k=1}^{\infty}$. It is clear that a frame sequence is a Bessel sequence.

Definition 3.3. *[18] Let $\psi \in L^2(\mathbb{R})$. For $j, k \in L^2(\mathbb{Z})$, define the function $\psi_{j,k}$ by*

$$\psi_{j,k} = D^j T_k \psi.$$

Then, we say the function ψ is a wavelet if the set $\left\{D^j T_k \psi\right\}_{j,k\in\mathbb{Z}}$ forms an orthonormal basis for $L^2(\mathbb{R})$.

The study of wavelet bases began around 1985, but the first example of wavelet is Haar wavelet which appeared much earlier. We will define the Haar wavelet later in Section 4.

Definition 3.4. *[4] Given $\Psi = \{\psi_\ell\}_{\ell=1}^{r} \subset L^2(\mathbb{R})$, define the affine system*

$$X(\Psi) = \{\psi_{\ell,j,k} : 1 \leq \ell \leq r \quad ; \; j,k \in \mathbb{Z}\} \subset L^2(\mathbb{R}),$$

where

$$\psi_{\ell,j,k} = D^j T_k \psi_\ell.$$

The system $X(\Psi)$ is called a tight wavelet frame of $L^2(\mathbb{R})$, if the following representation holds for all $f \in L^2(\mathbb{R})$. such that

$$\|f\|^2_{L^2(\mathbb{R})} = \sum_{g \in X(\Psi)} |\langle f, g\rangle|^2 ,$$

where $\langle \cdot, \cdot \rangle$ is the inner product in $L^2(\mathbb{R})$ and $\| \cdot \|_{L^2(\mathbb{R})} = \sqrt{\langle \cdot, \cdot \rangle}$. This is equivalent to say that

$$f = \sum_{g \in X(\Psi)} \langle f, g\rangle \, g, \;\; \textit{for all } f \in L^2(\mathbb{R}).$$

As we mentioned before, a tight frame is a generalization of an orthonormal basis, so its clear that an orthonormal basis is a tight frame. When $X(\Psi)$ is an orthonormal basis of $L^2(\mathbb{R})$, then we say that $X(\Psi)$ is an orthonormal wavelet basis.

The Fourier transform of a function $f \in L^2(\mathbb{R})$ is defined to be

$$\hat{f}(\omega) = \int_{\mathbb{R}} f(x) \; e^{-i\omega x} dx, \quad \omega \in \mathbb{R},$$

and the Fourier series of a sequence $h \in \ell_2(\mathbb{Z})$ is defined by

$$\hat{h}(\omega) = \sum_{g \in X(\Psi)} h[k] e^{-i\omega k}, \;\; \omega \in \mathbb{R}.$$

4 The Gibbs Effects in Orthogonal Expansions

In this section, we begin the study of Gibbs effects in the truncated orthogonal expansion of a specific function with jump discontinuities. Our example will cover the Bessel and the Legendre function series expansions.

The Gibbs effects in the orthogonal expansions was implicitly mentioned by Gottlieb and Orszag [19] in 1977 in a small chapter published for a survey in approximation theory, but they were not aware of this effect until late 1991. We will consider their setup in our presentation. Let us define some elements to represent the orthonormal series expansions.

Let $\{\varphi_n\}_{n=1}^{\infty}$ be a complete set of orthonormal functions defined on the interval (a, b), with the weighted inner product

$$\int_a^b \rho(x) \varphi_n(x) \overline{\varphi_m(x)} dx = \begin{cases} 0 & \text{if } n \neq m \\ 1 & \text{if } n = m \end{cases} ,$$

where $\rho(x)$ is the weight function and $\overline{\varphi_m(x)}$ indicates the complex conjugation of $\varphi_m(x)$. The orthonormal expansion of a function in terms of the set above is given by,

$$f(x) = \sum_{k=1}^{\infty} c_k \varphi_k(x) ,$$

where the coefficients c_k are

$$c_k = \int_a^b \rho(x)\ f(x)\,\overline{\varphi_k(x)}dx.$$

If the set $\{\varphi_n\}_{n=1}^{\infty}$ is orthogonal (but not orthonormal), i.e., $\|\varphi_n\| \neq 1$, for any n. Then

$$\int_a^b \rho(x)\varphi_n(x)\,\overline{\varphi_m(x)}dx = \begin{cases} 0 & \text{if } n \neq m \\ \|\varphi_n\|^2 & if \quad n = m \end{cases},$$

and

$$c_k = \frac{\int_a^b \rho(x)\ f(x)\,\overline{\varphi_k(x)}dx}{\|\varphi_k\|^2}.$$

4.1 The Bessel series expansion

The differential equation

$$\frac{d}{dx}\left(x\frac{du}{dx}\right) - \frac{n^2}{x}u(x) + \mu^2 xu(x) = 0, \quad 0 < x < a,$$

can be considered to generate the solution as an infinite series expansion about $x = 0$ by using the Frobenius method. This series solution give us the Bessel function $J_n(x)$ of the first kind of order $n \geq 0$ (n not necessarily an integer), which has the following popular representation,

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+n}}{2^{2k+n} k!\,(n+k)!}.$$

It is given that $u(a) = 0$, known as the vanishing boundary condition, which means

$$u(a) = J_n(j_{n,k}) = 0, \quad k = 1, 2, 3, \cdots,$$

where $\{j_{n,k}, k = 1, 2, \cdots\}$ are the zeros of the Bessel function $J_n(x)$.

4.1.1 Some properties of the Bessel functions.

For any $n > 0$, there are many properties of the Bessel functions [20], the most useful ones are:

- $J_n(-x) = (-1)^n J_n(x)$.
- $\frac{d}{dx}J_n(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$.
- $\frac{d}{dx}\left(x^n J_n(x)\right) = x^n J_{n-1}(x)$.
- $\frac{d}{dx}\left(x^{-n} J_n(x)\right) = -x^{-n} J_{n+1}(x)$.
- $J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$.

Definition 4.1. *The Bessel series expansion in general is given by,*

$$f(x) = \sum_{k=1}^{\infty} c_k J_n\left(\frac{j_{n,k}x}{a}\right), \quad 0 < x < a$$

where $f(x)$ defined on $(0, a)$ and,

$$c_k = \frac{\int_0^a x\ f(x)\ J_n\left(\frac{j_{n,k}x}{a}\right)dx}{\int_0^a x\ \left|J_n\left(\frac{j_{n,k}x}{a}\right)\right|^2 dx}.$$

Using the properties of Bessel functions, we have

$$\int_0^a x \left| J_n(\frac{j_{n,k}x}{a}) \right|^2 dx = \frac{a^2}{2} \left(J_{n+1}(j_{n,k})\right)^2 .$$

The truncated Bessel series of $f(x)$ of order N is given by,

$$S_N(x) = \sum_{k=1}^{N} c_k J_n(\frac{j_{n,k}x}{a}).$$

4.2 Some illustration

We give some simple illustrations of the Gibbs effects in the Bessel functions of the function $f(x)$, where

$$f(x) = \begin{cases} -2x & if \quad 0 < x < 1/2 \\ 2 - 2x & if \quad 1/2 < x < 1 \end{cases} .$$

Fig. 1 illustrates the Gibbs effects near the jump discontinuity point $x = 1/2$ in the domain $(0, 1)$. We used $N = 10, 20, 50$ terms in the N^{th} partial sum $S_N(x)$.

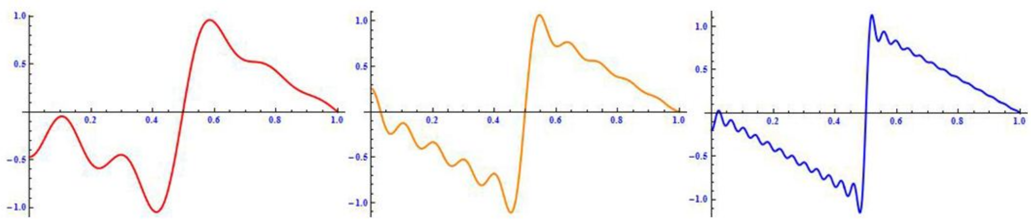

Fig. 1. The effects of the truncated Bessel series $S_N(x), N = 10, 20, 50$ with $f(x)$

Fig. 2 is obtained by showing all graphs together including f.

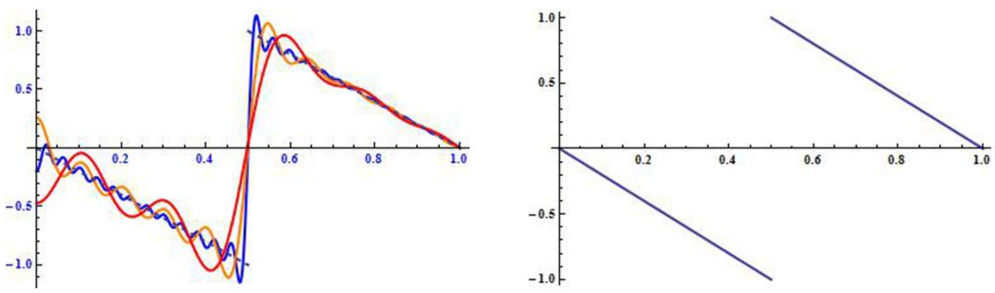

Fig. 2. The effects of the truncated Bessel series $S_N(x), N = 10, 20, 50$ with $f(x)$

4.3 The Legendre series expansion

One of the solutions of the differential equation

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{du}{dx}\right) + n(n+1)u(x) = 0, \qquad -1 < x < 1,$$

is the Legendre polynomial $P_n(x)$ which is a polynomial of degree n which can be expressed by the Rodrigues rule, namely,

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(1 - x^2\right)^n .$$

This family of polynomials is orthogonal on the interval $(-1, 1)$, such that $\|P_n\|^2 = \frac{2}{2n+1}$. More precisely,

$$\frac{2n+1}{2} \int_{-1}^{1} \rho(x) P_n\left(x\right) P_m\left(x\right) dx = \delta_{n,m},$$

where $\delta_{n,m}$ is the Kronecker delta function and $\rho(x) = 1$.

The Legendre series expansion in general is given by,

$$f\left(x\right) = \sum_{k=1}^{\infty} c_k P_k(x), \;\; for \; x \in (-1, 1),$$

where,

$$c_k = \frac{2n+1}{2} \int_{-1}^{1} f(x) \;\; P_n\left(x\right) dx.$$

The N-th partial sum of the Legendre series is defined by

$$S_N(x) = \sum_{k=0}^{N} \frac{(-1)^k \left(4k+3\right)! \left(2k\right)!}{2^{2k+1} \left(k+1\right)!} P_{2n+1}(x), \;\; -1 < x < 1.$$

4.4 Some illustration

We give some simple illustrations of the Gibbs effects for the Legendre series approximation of the wave (square) function $g(x)$, where

$$g\left(x\right) = \left\{ \begin{array}{ll} -1 & if \; 0 \leq x < 1/2 \\ 1 & if \; 1/2 < x < 1 \end{array} \right. .$$

Fig. 3 illustrates the Gibbs effects near the jump discontinuity point $x = 1/2$ of the domain $(0, 1)$. We used $N = 10, 20, 50$ terms in the Nth partial sum $S_N\left(x\right).$

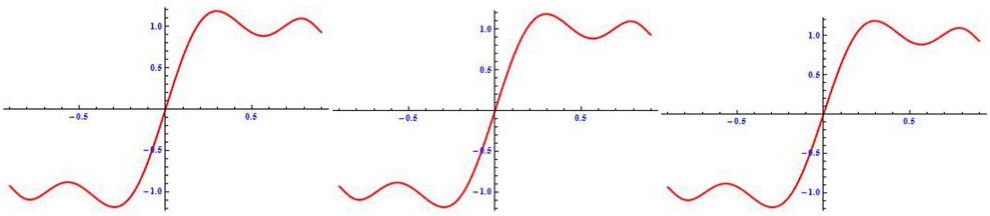

Fig. 3. The effects of the truncated Legendre series $S_N(x), N = 10, 20, 50$

Fig. 4 is obtained by showing all graphs together including g.

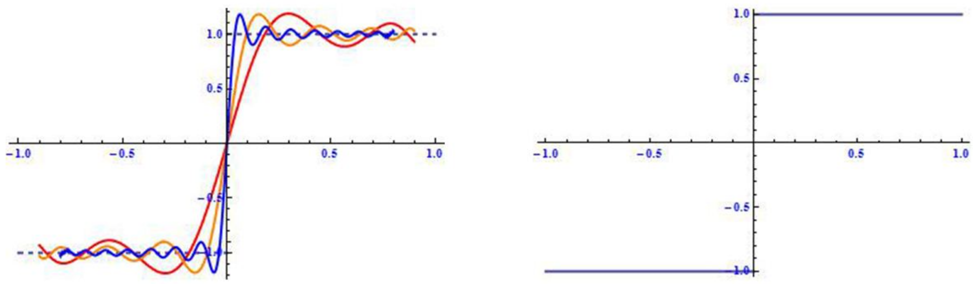

Fig. 4. The effects of the truncated Bessel series $S_N(x), N = 10, 20, 50$ with $f(x)$

5 MRA, FMRA and Framelets Generated by the UEP

Multiresolution analysis (MRA) is a tool to construct orthonormal bases for $L^2(\mathbb{R})$ of the form $D^jT_k\psi; j,k \in \mathbb{Z}$ for a suitably chosen function $\psi \in L^2(\mathbb{R})$. Such a function is called a *wavelet.* The introduction of MRA by Mallat and Meyer [21, 22] was the beginning of a new era, in fact the concept of a multiresolution analysis is fundamental in wavelet theory. It consists of a collection of conditions on certain subspaces of $L^2(\mathbb{R})$.

Frame MRA was introduced by Benedetto and Li [23, 24]. The theory provides conceptually new mathematical and signal processing results, which go beyond the simple combination of frame and MRA techniques.

Definition 5.1. *A frame multiresolution analysis (FMRA) $\{V_j, \phi\}_{j\in\mathbb{Z}}$ for $L^2(\mathbb{R})$ is a sequence of closed subspaces $V_j \subseteq L^2(\mathbb{R})$ and an element $\phi \in V_0$ such that:*

(i) $\cdots V_{-1} \subset V_0 \subset V_1 \cdots$, *(Nested sequence).*

(ii) $\overline{\cup_j V_j} = L^2(\mathbb{R})$ *and* $\cap_j V_j = 0$, *(Density and Separation).*

(iii) $f \in V_0 \Longrightarrow T_k f \in V_0$, $\forall k \in \mathbb{Z}$, *(Translation).*

(iv) $V_{j+1} = D(V_j)$, *i.e.,* $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}$, *(Scaling).*

(v) $\{T_k\phi\}_{k\in\mathbb{Z}}$ *is a frame for* V_0.

The function ϕ is called the generator of the frame multiresolution analysis. Alternatively, a FMRA is defined as a MRA, with the condition (v) and the orthonormal basis for V_0 replaced by a frame condition. So we can start the construction of a FMRA with a subspace $V_0 \subset L^2(\mathbb{R})$ satisfying conditions (i)-(iii) and then try to find a function ϕ such that $\{T_k\phi\}_{k\in\mathbb{Z}}$ is a frame for V_0.

Example 5.1. *(Haar MRA) We can define a MRA by,*

$$\begin{aligned} \phi &= 1_{[0,1)}; \\ V_j &= \left\{ f \in L^2(\mathbb{R}) : f \text{ is constant on } [2^{-j}k, 2^{-j}(k+1)],\ \forall k \in \mathbb{Z} \right\}. \end{aligned}$$

Note that the Haar wavelet can be written as,

$$\begin{aligned} \psi(x) &= 1_{[0,1/2)}(x) - 1_{[1/2,1)}(x) \\ &= 1_{[0,1)}(2x) - 1_{[0,1)}(2x-1) \\ &= \frac{1}{\sqrt{2}}\left(D1_{[0,1)}(x) - DT_1 1_{[0,1)}(x) \right). \end{aligned}$$

Fig. 5 is for the graph of the Haar scaling function with its wavelet.

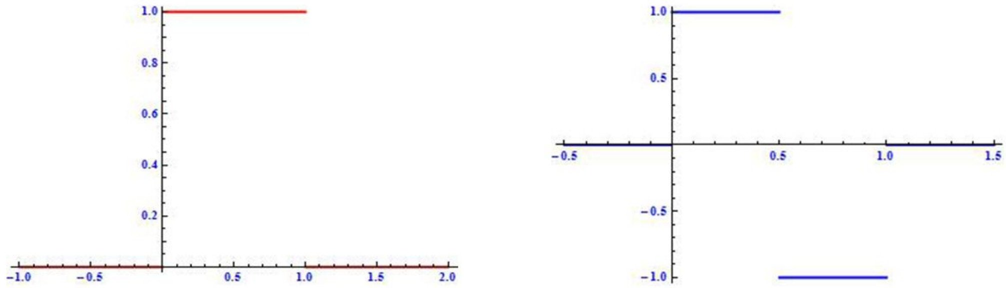

Fig. 5. The scaling function ϕ with its wavelet ψ

6 Unitary Extension Principle (UEP)

The Unitary Extension Principle is a principle for construction a tight frame for $L^2(\mathbb{R})$. We will recall a result by Ron and Shen, which constructs tight wavelet frames from the UEP generated by a collections $\{\psi_\ell\}_{\ell=1}^r$ called framelet.

Definition 6.1. *A compactly supported function $\phi \in L^2(\mathbb{R})$ is said to be refinable if*

$$\phi(x) = 2\sum_{k\in\mathbb{Z}} h_0[k]\ \phi(2x-k), \tag{6.1}$$

for some finite supported sequence $h_0[k] \in \ell_2(\mathbb{Z})$. The sequence h_0 is called the low pass filter of ϕ.

The general setup is to construct tight frame for $L^2(\mathbb{R})$ of the form

$$\{\psi_{\ell,j,k} : 1 \le l \le r \ \ ;\ j,k \in \mathbb{Z}\}$$

which can be summarized as follows: Let V_0 be the closed space generated by $\{T_k\phi\}_{k\in\mathbb{Z}}$, i.e., $V_0 = \overline{span}\,\{T_k\phi\}_{k\in\mathbb{Z}}$, and $V_j = \left\{\ f(2^j x) : f(x) \in V_0 \text{ where } x \in \mathbb{R}\right\}$. Let $\{V_j,\phi\}_{j\in\mathbb{Z}}$ be the MRA generated by the function ϕ and $\Psi = \{\psi_\ell\}_{\ell=1}^r \subset V_1$ such that

$$\psi_\ell = 2\sum_{k\in\mathbb{Z}} h_\ell[k]\ \phi(2\cdot -k), \tag{6.2}$$

where$\{h_\ell[k],\ k \in \mathbb{Z}\}_{\ell=1}^r$ is a finitely supported sequences and called *high pass filters* of the system.

Theorem 6.1. *(Unitary Extension Principle)[25] Let $\phi \in L^2(\mathbb{R})$ be the compactly supported refinable function with its finitely supported low pass filter h_0. Let*

$$\{h_\ell[k],\ k \in \mathbb{Z}\}_{\ell=1}^r$$

be a set of finitely supported sequences, then the system

$$X\left(\Psi\right) = \{\psi_{\ell,j,k} :\quad 1 \le \ell \le r \ \ ; j,k \in \mathbb{Z}\} \tag{6.3}$$

forms a tight frame for $L^2(\mathbb{R})$ provided the equalities

$$\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}} \overline{h_\ell[k]}h_\ell[k-p] = \delta_{0,p} \tag{6.4}$$

and

$$\sum_{\ell=0}^{r}\sum_{k\in\mathbb{Z}}(-1)^{k-p}\,\overline{h_\ell[k]}h_\ell[k-p]=0 \tag{6.5}$$

hold for all $p\in\mathbb{Z}$, *where* $\delta_{0,p}=\begin{cases}0, & if\ \ p\neq 0\\ 1, & if\ p=0\end{cases}$. *Particularly, if* $r=1$ *and* $\|\phi\|_2=1$, *then* $X(\Psi)$ *is an orthonormal wavelet bases of* $L^2(\mathbb{R})$.

For the proof, please see [25]. By looking to the proof of Theorem 6.1, we have the following *framelet representation*,

$$f=\sum_{\ell=1}^{r}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\langle f,\psi_{\ell,j,k}\rangle\,\psi_{\ell,j,k}. \tag{6.6}$$

For a given ϕ, we define the *projection operator* by

$$P_j:\ f\mapsto\sum_{k\in\mathbb{Z}}\langle f,\phi_{j,k}\rangle\ \ \phi_{j,k},$$

for any $f\in L^2(\mathbb{R})$, where $\phi_{j,k}=D^jT_k\phi$. Since $\phi\in V_0$, its clear that $P_jf\in V_j$. By the refinablity of ϕ we have

$$\sum_{k\in\mathbb{Z}}\phi(x+k)=1,$$

and,

$$\lim_{j\to\infty}P_jf=f\ \ \left(or\ \|P_jf-f\|_{L^2(\mathbb{R})}\longrightarrow 0\ as\ j\longrightarrow\infty\right).$$

We provide a precise definition of the Gibbs effects for framelet expansion of a given function $f:\mathbb{R}\longrightarrow\mathbb{R}$, such that $f\in L^2(\mathbb{R})$ has a compact support and a jump discontinuity at the origin as follows.

Definition 6.2. *Suppose a function* f *has a jump discontinuity at* $x=0$, *i.e., limits* $\lim_{x\to0^+}f(x)$ *and* $\lim_{x\to0^-}f(x)$ *exist, and that* $f\left(0^+\right)\neq f\left(0^-\right)$. *Define* $S_{n,m}f$ *to be the truncated partial sum of Equation (6.6) where* j *and* k *are up to* n *and* m *respectively. We say that the framelet expansion of* f *exhibits the Gibbs effects at the right hand side of* $x=0$ *if there is a sequence* $x_s>0$ *converging to* 0, *and*

$$\lim_{n,m\to\infty}S_{n,m}f(x_s)\begin{cases}>f(0^+), & if\quad f\left(0^+\right)>f\left(0^-\right)\\ <f(0^+), & if\quad f\left(0^+\right)<f\left(0^-\right)\end{cases}.$$

Similarly, we can define the Gibbs effects on the left-hand side of $x=0$.

7 Quasi-affine Systems

The notion of quasi-affine systems was first introduced in [26]. We present a quasi-affine system that allows us to construct a tight framelet that is not an orthonormal basis. We use the Haar scaling function to generate a quasi-affine system to investigate the Gibbs effects.

Definition 7.1. *Let* Ψ *be defined as in the UEP. The corresponding quasi-affine system* $X^J(\Psi)$ *generated by* Ψ *is defined by a collection of translations and dilations of the elements in* Ψ *such that*

$$X^J(\Psi)=\{\psi_{\ell,j,k}:1\leq\ell\leq r,j,k\in\mathbb{Z},\}$$

where,

$$\psi_{\ell,j,k}=\begin{cases}2^{j/2}\psi_\ell(2^j\cdot-k), & j\geq J\\ 2^j\psi_\ell(2^j(\cdot-k)), & j<J\end{cases}.$$

When $X^J(\Psi)$ forms a tight frame of $L^2(\mathbb{R})$, we call the function ψ_ℓ a tight framelet and $X^J(\Psi)$ a tight quasi-affine framelet system. With this definition one can see that the quasi-affine system is constructed by changing the basic definition of $\psi_{\ell,j,k}$ in the UEP, i.e. by sampling the wavelet frame system starting from the level $J-1$ and downward. The framelet representation will not change with this definition, however, we have a tight frame instead of the orthonormal system [25].

In the study of our expansion, we consider $J=0$ where

$$X^0(\Psi) = \{\psi_{\ell,j,k} : j,k \in \mathbb{Z}\}$$

generates a tight frame for $L^2(\mathbb{R})$ but not an orthonormal basis. If we consider ψ_m, then this system uses the B-splines scaling functions to construct the generator $\psi_{\ell,j,k}$, and we refer to it as the B-splines tight framelet system.

8 Construction of B-splines Tight Framelet Using UEP

The choice of the number r for Ψ in the UEP, requires a little exertion to construct the framelets. We will give some examples of constructing B-spline tight framelet.

Definition 8.1 (**B-splines**). *The B-spline N_{m+1} is defined as follows by using the convolution*

$$N_{m+1}(x) := (N_m * N_1)(x), x \in \mathbb{R} \tag{8.1}$$

where, $N_1(x)$ is defined to be $\chi_{[0,1)}(x)$ (the characteristic function for the interval $[0,1)$).

The B-spline N_m is said to have order m. By definition of the convolution, we have that

$$N_{m+1}(x) = \int_0^1 N_m(x-t)\,dt.$$

The B-splines are well studied, and have many desirable properties [27]. Note that N_m is symmetric with respect to the center of its support: $N_m(\frac{m}{2}+x) = N_m(\frac{m}{2}-x)$. $N_m(x)$ has the following dyadic dilation relation

$$N_m(x) = \sum_{k=0}^{m} 2^{-m+1}\binom{m}{k} N_m(2x-k), \tag{8.2}$$

where

$$\binom{m}{k} = \frac{m!}{k!\,(m-k)!}.$$

Explicit formula for the B-spline is given by

$$N_m(x) = \frac{1}{(m-1)!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\ (x-k)_+^{m-1},$$

where

$$(x-k)_+^{m-1} = (\max(0, x-k))^{m-1},\ m = 2,3,4,\cdots.$$

Definition 8.2 (**The Centered B-splines [28]**). *For $m \in \mathbb{N}$, the centered B-spline B_m is defined by*

$$B_m(x) := T_{-\frac{m}{2}} N_m(x).$$

Alternatively, the B-spline B_m can be defined by

$$B_1(x) = \chi_{[-1/2,1/2]}(x), B_{m+1} := B_m * B_1, m \in \mathbb{N}.$$

It is clear that this definition leads to the same functions. Thus, for any $m \in \mathbb{N}$, we have:

$$B_{m+1}(x) = \int_{-1/2}^{1/2} B_m(x-t)\,dt.$$

Note that an explicit expression for B_m can be obtained easily from the explicit expression of N_m above.

Lemma 8.1. *The low pass filter $h_0[k]$ of N_m is equal to $2^{-m}\binom{m}{k}$.*

Proof. Obvious by looking to Equation (8.2) and Definition 6.1 □

Example 8.1. *Let $\phi(x) = N_2(x)$ be the piecewise linear function (the hat function) defined on $[0,2]$, we have*

$$N_2(x) = N_1 * N_1(x) = \int_0^1 N_1(x-t)\,dt.$$

Then,

$$N_2(x) = \begin{cases} x & \text{if } 0 \le x \le 1 \\ 2-x & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}.$$

Similar calculation, from Equation 8.1, we can find the quadratic and cubic piecewise B-splines $N_3(x)$ and $N_4(x)$, respectively :

$$N_3(x) = \begin{cases} \frac{1}{2}x^2 & \text{if } 0 \le x \le 1 \\ -x^2+3x-\frac{3}{2} & \text{if } 1 \le x \le 2 \\ \frac{1}{2}x^2-3x+\frac{9}{2} & \text{if } 2 \le x \le 3 \\ 0 & \text{if } x<0 \text{ and } x>3 \end{cases},$$

$$N_4(x) = \frac{1}{6}\begin{cases} x^3 & \text{if } 0 \le x \le 1 \\ 4-6(x-2)^2-3(x-2)^3 & \text{if } 1 \le x \le 2 \\ 4-6(x-2)^2+3(x-2)^3 & \text{if } 2 \le x \le 3 \\ -(x-4)^3 & \text{if } 3 \le x \le 4 \\ 0 & \text{if } x \in \mathbb{R}-[0,4] \end{cases}$$

Fig. 6 shows these B-splines of order 2 and 4.

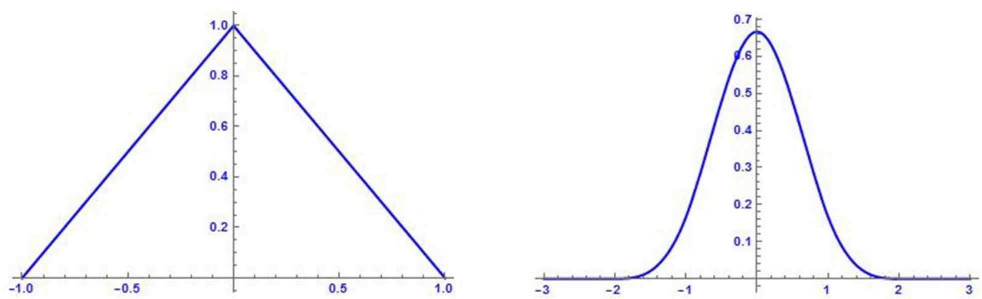

Fig. 6. The B-splines of order 2 and 4

The absence of Gibbs effect has been examined and proven in [5] for fixed number of generators. In this paper, we will use different number of generators by taking $r = 3$. Then we will have different generators for the space $L^2(\mathbb{R})$. Let's consider the B-spline of order 2 and 4 to find a set of three generators.

As an application, we present a construction of compactly supported B-spline framelet.

Example 8.2. *The linear B-spline* $B_2(x)$ *when* $r = 3$ *has three high pass filters,* $\{h_1[k], h_2[k], h_3[k], k \in \mathbb{Z}\}$. *Let* $h_0 = [\frac{1}{4}, \frac{1}{2}, \frac{1}{4}]$ *be the low pass filter of* $B_2(x)$. *Consider the system*

$$\Psi = \{\psi_\ell\}_{\ell=1}^{3}$$

to be defined as Theorem 6.1. Now we need to find

$$\{h_\ell[k],\ k \in \mathbb{Z}\}_{\ell=1}^{2}$$

such that Equation (6.4) and Equation (6.5) hold. This generates the following nonlinear system of equations:

$$\begin{cases} \sum_{k\in\mathbb{Z}} \left(h_1^2[k] + h_2^2[k] + h_3^2[k]\right) = d_1, \\ \sum_{k\in\mathbb{Z}} \left(h_1[k]h_1[k-1] + h_2[k]h_2[k-1] + h_3[k]h_3[k-1]\right) = d_2, \\ \sum_{k\in\mathbb{Z}} \left(h_1[k]h_1[k-2] + h_2[k]h_2[k-2] + h_3[k]h_3[k-2]\right) = d_3, \\ \sum_{k\in\mathbb{Z}} \left(h_1[k]h_1[k-3] + h_2[k]h_2[k-3] + h_3[k]h_3[k-3]\right) = d_4, \\ \sum_{k\in\mathbb{Z}} (-1)^k \left(h_1^2[k] + h_2^2[k] + h_3^2[k]\right) = d_5, \\ \sum_{k\in\mathbb{Z}} (-1)^{k-1} \left(h_1[k]h_1[k-1] + h_2[k]h_2[k-1] + h_3[k]h_3[k-1]\right)) = d_6, \\ \sum_{k\in\mathbb{Z}} (-1)^{k-2} \left(h_1[k]h_1[k-2] + h_2[k]h_2[k-2] + h_3[k]h_3[k-2]\right)) = d_7, \\ \sum_{k\in\mathbb{Z}} (-1)^{k-2} \left(h_1[k]h_1[k-3] + h_2[k]h_2[k-3] + h_3[k]h_3[k-3]\right)) = d_8, \end{cases}$$

for some constants $d_i, i = 1, \cdots, 8$. *We use MATHEMATICA software and obtain different choices of solutions for* h_1, h_2, *and* h_3, *namely,*

$$\begin{aligned} h_1 &= [\frac{-\sqrt{2}}{16}, \frac{-\sqrt{2}}{8}, \frac{3\sqrt{2}}{8}, \frac{-\sqrt{2}}{8}, \frac{-\sqrt{2}}{16}], \\ h_2 &= [\frac{-\sqrt{2}}{8}, \frac{\sqrt{2}}{4}, \frac{-\sqrt{2}}{8}], \\ h_3 &= [\frac{-\sqrt{2}}{16}, \frac{-\sqrt{2}}{8}, \frac{\sqrt{2}}{8}, \frac{\sqrt{2}}{16}]. \end{aligned}$$

Using Theorem 6.1 together with the filters $\{h_\ell[k],\ k \in \mathbb{Z}\}_{\ell=1}^{3}$ *and* $B_2(x)$, *then, the corresponding* $X(\{\psi_\ell\}_{\ell=1}^{3})$ *generates a tight framelet for* $L^2(\mathbb{R})$.

The graphs of the generators are shown in Fig. 7.

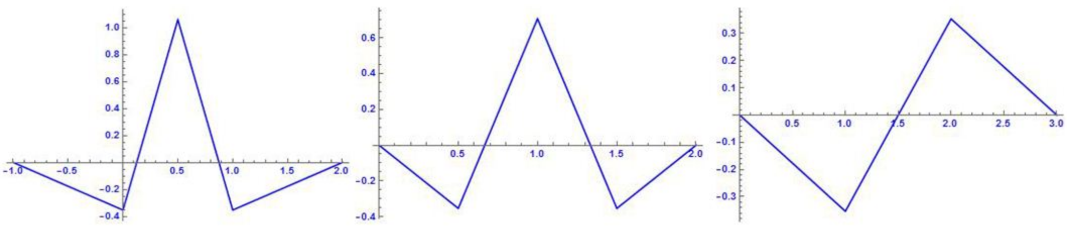

Fig. 7. The generators using, $\psi_1(x)$**,** $\psi_2(x)$ **and** $\psi_3(x)$**,** $B_2(x)$

Example 8.3. *For* $B_4(x)$, *the cubic B-spline, and when* $r = 3$, *then we have the B-spline tight framelets* $\{\psi_\ell\}_{\ell=1}^{3}$. *Let* $h_0 = [\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}]$ *be the low pass filter of* $B_4(x)$.

Define

$$
\begin{aligned}
h_1 &= [\frac{-\sqrt{2}}{256}, \frac{-\sqrt{2}}{64}, \frac{-3\sqrt{2}}{64}, \frac{-7\sqrt{2}}{64}, \frac{45\sqrt{2}}{128}, \frac{-7\sqrt{2}}{64}, \frac{-3\sqrt{2}}{64}, \frac{-\sqrt{2}}{64}, \frac{-\sqrt{2}}{256}],\\
h_2 &= [\frac{-\sqrt{2}}{64}, \frac{-\sqrt{2}}{16}, \frac{-7\sqrt{2}}{64}, \frac{3\sqrt{2}}{8}, \frac{-7\sqrt{2}}{64}, \frac{-\sqrt{2}}{16}, \frac{-\sqrt{2}}{64}],\\
h_3 &= [(\frac{1}{64} + \frac{\sqrt{14}}{256}), (\frac{1}{16} + \frac{\sqrt{14}}{64}), (\frac{3}{32} + \frac{\sqrt{14}}{64}), (\frac{1}{16} - \frac{\sqrt{14}}{64}), -\frac{5\sqrt{14}}{128}, -(\frac{1}{16} + \frac{\sqrt{14}}{64}),\\
&\quad -(\frac{3}{32} - \frac{\sqrt{14}}{64}), (\frac{-1}{16} + \frac{\sqrt{14}}{64}), -(\frac{1}{64} - \frac{\sqrt{14}}{256})].
\end{aligned}
$$

Then, h_0, h_1, h_2, and h_3 satisfied by Equations 6.3 and 6.4. Hence, the system $X(\Psi)$ is a tight framelet of $L^2(\mathbb{R})$. Again, this is due to Theorem 6.1.

We illustrate the generators ψ_1, ψ_2, and ψ_3 in Fig. 8.

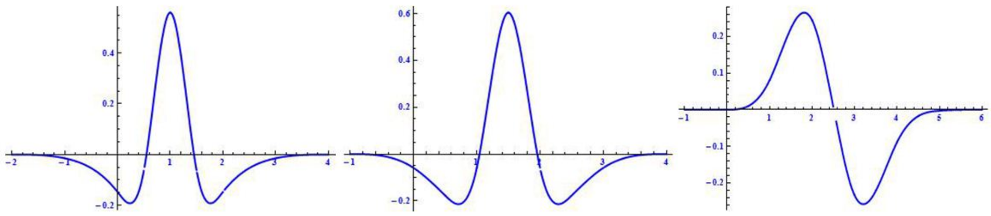

Fig. 8. The generators, $\psi_1(x)$, $\psi_2(x)$ and $\psi_3(x)$, using $B_4(x)$

9 The Gibbs Effects in B-spline Tight Framelets Using Three Generators

It is well known that the Fourier series of a function with jump discontinuity exhibits Gibbs effects [20]. The partial sums overshoot the function near the discontinuity, and this overshoot continues no matter how many terms are taken in the partial sum. More precisely, if f is a piecewise Lipschitz continuous function with a (positive) jump discontinuity at $x = a$, then there is a sequence $x_n > a \to a^+$ such that

$$\lim_{n\to\infty} S_n(x) > f(a^+),$$

where S_n is the sequence of partial sums of the Fourier series. Gibbs effects does not occur if the partial sums replaced by the average of the partial sums $\sigma_n(x) = \frac{S_n(x)}{n}$, [29].

It has been shown in [2] that all standard wavelet expansions do exhibit this overshoot effects at the origin. Also, in [5] similar results for B-spline framelets has been examined, we will present similar numerical result for B-splines tight framelets constructed from the quasi-affine system using UEP and with a fixed number of generators, $r = 3$.

Definition 9.1. *For any function $f \in L^2(\mathbb{R})$, f can be expanded by:*

$$f = \sum_{\ell=1}^{r} \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \left\langle f, D^j T_k \psi_\ell \right\rangle \; D^j T_k \psi_\ell.$$

The standard approximation is the series given by,

$$S^r_{n,m} f = \sum_{\ell=1}^{r} \sum_{j<m} \sum_{k=-n}^{n} \langle f, \psi_{\ell,j,k} \rangle \, \psi_{\ell,j,k}.$$

Consider the B-spline, B_M, for fixed $M \geq 2$. Then, we have the truncated B-spline tight framelet expansion given by,

$$S_n^{r,M} f(\cdot) = \int_{\mathbb{R}} f(y) K_n^{r,M}(x,y)\, dy,$$

where

$$K_n^{r,M}(x,y) = \sum_{\ell=1}^{r} \sum_{j} \sum_{k} \psi_{\ell,j,k}^{M}(x)\, \psi_{\ell,j,k}^{M}(y).$$

In what follows, we study the Gibbs effects of functions that have a jump discontinuity at 0, by considering the following function,

$$f(x) = \begin{cases} 1-x, & 0 < x \leq 1 \\ -1-x, & -1 \leq x < 0 \\ 0, & \text{else} \end{cases}.$$

Changing the number of generators for the B-spline tight framelet will not change the absence of Gibbs effects. When we changed the number of generators we found different tight framelets for the linear B-spline tight framelet of order 2 and 4. However, the numerical errors were very close to each other. Table 1 and Table 2 showing an approximation of the Gibbs effects (by finding the maximum overshoots and undershoots) for $S_n^{r,2} f$, $S_n^{r,4} f$, when $r = 3$ at the neighborhood of the jump $x = 0$.

Table 1. Approximate maximum overshoot and undershoot in neighborhoods of $x = 0$ using $S_n^{3,2} f$

Level	Maximum	Minimum
$n = 1$	0.693359	-0.703125
$n = 5$	0.984375	-0.984375
$n = 10$	0.999512	-0.999512
$n = 15$	0.999998	-0.999998
$n = 20$	≈ 1	≈ -1

Table 2. Approximate maximum overshoot and undershoot in neighborhoods of $x = 0$ using $S_n^{3,4} f$

Level	Maximum	Minimum
$n = 1$	0.38512	-0.125626
$n = 5$	0.959504	-0.959401
$n = 10$	0.998570	-0.998570
$n = 20$	≈ 1	≈ -1

Fig. 9 illustrates the absence of the Gibbs effects for a set of three generators using the linear B-spline tight framelet.

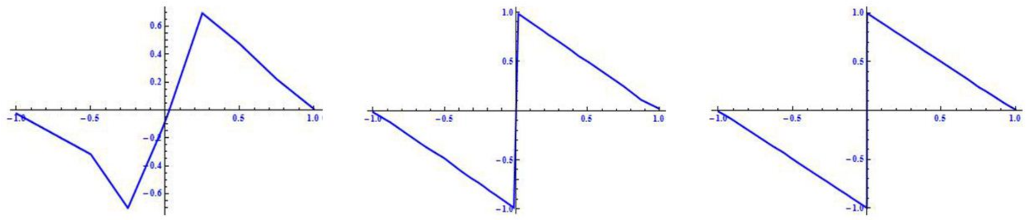

Fig. 9. Illustration for the non-overshoot and non-undershoot of the $S_n^{3,2} f(x)$

Fig. 10 illustrates the absence of the Gibbs effects for a set of three generators using the cubic B-spline tight framelet.

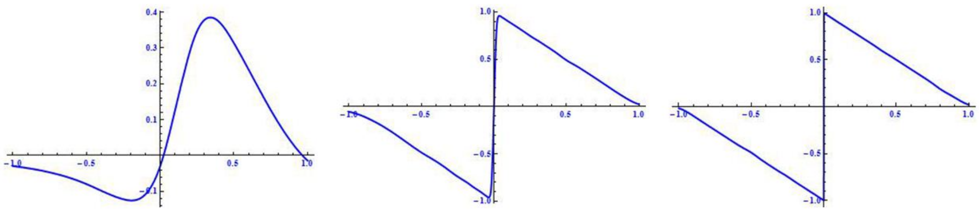

Fig. 10. Illustration for the non-overshoot and non-undershoot of the $S_n^{3,4} f(x)$

10 Conclusion

The method of numerical approximation is a rich subject. It gives a powerful tool and has many applications. Here, we have presented our numerical approximation method by using framelet representations. This not only gives a new way to do approximation but also show many options and useful methods to do applications. Some features of the given object may not be discovered by using orthogonal expansions, alternatively, the redundant systems may help to keep the originality of the object. It is anticipated this new method will be applied to many applications in various areas.

It is known that the key property of frames is the possibility of redundancy. Actually it can be a useful and even essential property in many settings and have found many practical applications both in mathematics and engineering. For example, the redundant system offered by framelets has already been put to good use for signal denoising, in the context of signal transmission, and image compression [25]. The redundant system leads to a reduction in the inevitable *quantization error* [18], which appears in all applications of series representations. Furthermore, If we have this redundancy that built into the coefficients in the representation, then we might still be able to reconstruct the function (signal, image) f well from the remaining coefficients.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Weierstrass K. Uber die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen. Sitzungsberichte der Königlich Preubischen Akademie der Wissenschaften zu Berlin; 1885.

[2] Kelly S. Gibbs phenomenon for wavelets. Appl. Comp. Harm. Anal. 1996;3:72-81.

[3] Shim HT, Volkmer H. On Gibbs phenomenon for wavelet expansions. J. Approx. 1996;84:74-95.

[4] Bin Han. Framelets and wavelets: Algorithms, analysis, and applications. Applied and Numerical Harmonic Analysis, Birkhauser/Springer, Cham. 2017;724.

[5] Mohammad M, Lin EB. Gibbs phenomenon in B-spline tight framelet expansions. Communications in Nonlinear Science and Numerical Simulation. 2018;55:84-92.

[6] Korner TW. Fourier analysis. Cambridge University Press. 1989;62-66.

[7] Proakis JG, Dimitris GM. Introduction to digital signal processing. Macmillan Publishing Company, New York. 1988;551-559.

[8] Gottlieb D, Chi-Wang Shu. On the Gibbs Phenomenon and its resolution. SIAM Review. 1997;39(4):644-668.

[9] Petersen TE. Eliminating Gibbs effect from separation of variables solutions. SIAM Review. 1998;40(2):324-326.

[10] Fan SC, Zheng DY, Au FTK. Gibbs-phenomenon-free Fourier series for vibration and stability of complex beams. AIAA Journal. 2001;39(10):1977-1984.

[11] Li Q, Chen Y. Pseudo-spectral time-domain analysis using an initial-condition excitation technique for elimination of Gibbs phenomenon. Chinese Journal of Electronics. 2000;9(1):92-95.

[12] Monther Aldwairi, Yahya Flaifel. Baeza-Yates and navarro approximate string matching for spam filtering. In Proc. of the Second International Conference on Innovative Computing Technology (INTECH 2012), Rabat, Morocco, September 18-20, 2012

[13] Monther Aldwairi, Rami Alsalman. MALURLs: Malicious URLs classification system. In Proc. of the Annual International Conference on Information Theory and Applications, Singapore, Feb 28, 2011.

[14] Zhang Z, Martin CF. Convergence and Gibbs phenomenon in cubic spline interpolation of discontinuous functions. Journal of Computational and Applied mathematics. 1997;87(2):359-371.

[15] Njock J. Gibbs phenomenon in engineering systems, paper presented at 2005 Annual Conference, Portland, Oregon.

[16] Duffin RJ, Schaeffer AC. A class of nonharmonic Fourier series, Trans. Amer. Math. Soc. 1952;72:341-366.

[17] Daubechies I, Grossmann A, Meyer Y. Painless nonorthogonal expansions. J. Math. Phys. 1986;27:1271-1283.

[18] Christensen O. An introduction to frames and Riesz bases, Birkhauser, Boston; 2003.

[19] Gottlieb D, Orszag S. Numerical analysis of spectral methods: Theory and applications. Society for Industrial and Applied Mathematics; 1977.

[20] Jerri AJ. The Gibbs phenomenon in Fourier analysis, Splines and wavelet approximations. Kluwer Academic Publishers, Netherlands; 1998.

[21] Mallat SG. Multiresolution approximations and wavelet orthonormal bases of $L^2(\mathbf{R})$. Transactions of the American Mathematical Society. 1989;315(1):69–87.

[22] Meyer Y. Wavelets and operators. Translated by DH Salinger, Cambridge Studies in Advanced Mathematics; 1992.

[23] Benedetto J. The theory of multiresolution analysis frames and applications to filter banks. Applied and Computational Harmonic Analysis. 1998;5(4):389-427.

[24] Benedetto J, Li S. Subband coding and noise reduction in multiresolution analysis frames. Proc. SPIE 2303, wavelet Applications in Signal and Image Processing II; 1994.

[25] Daubechies I, Han B, Ron A, Shen Z. Framelets: MRA-based constructions of wavelet frames. Appl. Comput. Harmon. Anal. 2003;14:1-46.

[26] Ron A, Shen Z. Affine systems in $L^2(\mathbb{R}^d)$: the analysis of the analysis operators. J. Funct. Anal. 1997;148:408-447.

[27] Walnut D: An introduction to wavelet analysis. Birkhauser, Boston; 2001.

[28] Daubechies I. Ten lectures on wavelets. SIAM, Philadelphia; 1992.

[29] Zygmund A. Trigonometric series, 2nd ed., Cambridge Univ. Press, Cambridge; 1959.

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