Open Journal of
Discrete Applied Mathematics

PSR Press

Article

Some properties of the solutions of the difference equation $x_{n+1} = ax_n + \frac{bx_nx_{n-4}}{cx_{n-3} + dx_{n-4}}$

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Received: 23 January 2019; Accepted: 22 February 2019; Published: 9 July 2019.

Abstract: In this article, we study some properties of the solutions of the following difference equation: $x_{n+1} = ax_n + \frac{bx_nx_{n-4}}{cx_{n-3} + dx_{n-4}}, \quad n = 0, 1, ...$ where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers and a, b, c, d are positive constants. Also, we give specific form of the solutions of four special cases of this equation.

Keywords: Difference equations, recursive sequences, stability, boundedness.

MSC: 39A10.

1. Introduction

Our aim in this paper is to investigate the behavior of the solution of the following nonlinear difference equation

$$x_{n+1} = ax_n + \frac{bx_nx_{n-4}}{cx_{n-3} + dx_{n-4}}, \quad n = 0, 1, \ldots \tag{1}$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary positive real numbers and a, b, c and d are positive constants.
Recently there has been a great interest in studying the qualitative properties of rational difference equations. Some prototypes for the development of the basic theory of the global behavior of nonlinear difference equations of order greater than one come from the results for rational difference equations [1–15] .

However, there have not been any effective general methods to deal with the global behavior of rational difference equations of order greater than one so far. From the known work, one can see that it is extremely difficult to understand thoroughly the global behaviors of solutions of rational difference equations although they have simple forms (or expressions). One can refer to [16–47] for examples to illustrate this. Therefore, the study of rational difference equations of order greater than one is worth further consideration.

Many researchers have investigated the behavior of the solution of difference equations, for example, Elsayed *et al.* [37] has obtained results concerning the dynamics and global attractivity of the rational difference equation

$$x_{n+1} = \frac{ax_nx_{n-2}}{bx_{n-2} + cx_{n-3}}.$$

Aloqeili [18] has obtained the solutions of the difference equation

$$x_{n+1} = \frac{x_{n-1}}{a - x_nx_{n-1}}.$$

Simsek *et al*. [43] obtained the solution of the difference equation

$$x_{n+1} = \frac{x_{n-3}}{1 + x_{n-1}}$$

Çinar [22–24] got the solutions of the following difference equations

$$x_{n+1} = \frac{x_{n-1}}{1 + ax_n x_{n-1}}, x_{n+1} = \frac{x_{n-1}}{-1 + ax_n x_{n-1}}, x_{n+1} = \frac{ax_{n-1}}{1 + bx_n x_{n-1}}.$$

In [48], Ibrahim got the form of the solution of the rational difference equation

$$x_{n+1} = \frac{x_n x_{n-2}}{x_{n-1}(a + bx_n x_{n-2})}.$$

Karatas *et al*. [46] got the solution of the difference equation

$$x_{n+1} = \frac{x_{n-5}}{1 + x_{n-2} x_{n-5}}.$$

Here, we recall some notations and results which will be useful in our investigation. Let I be some interval of real numbers and let

$$f : I^{k+1} \quad I,$$

be a continuously differentiable function. Then for every set of initial conditions $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \quad n = 0, 1, \ldots \tag{2}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1. A point $\bar{x} \in I$ is called an *equilibrium point* of Equation (2) if $\bar{x} = f(\bar{x}, \bar{x}, ..., \bar{x})$. That is, $x_n = \bar{x}$ for $n \geq 0$, is a solution of Equation (2), or equivalently, $\bar{x}$ is a fixed point of f.

Definition 2.
- The equilibrium point $\bar{x}$ of Equation (2) is locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

we have $|x_n - \bar{x}| < \varepsilon$, for all $n \geq -k$.
- The equilibrium point $\bar{x}$ of Equation (2) is locally asymptotically stable if $\bar{x}$ is locally stable solution of Equation (2) and there exists $\gamma > 0$, such that for all $x_{-k}, x_{-k+1}, x_{-k+2}, ..., x_0 \in I$, with

$$|x_{-k} - \bar{x}| + |x_{-k+1} - \bar{x}| + |x_{-k+2} - \bar{x}| + ... + |x_0 - \bar{x}| < \delta,$$

we have $\lim_{n\infty} x_n = \bar{x}$.
- The equilibrium point $\bar{x}$ of Equation (2) is global attractor if for all $x_{-k}, x_{-k+1}, ..., x_0 \in I$ we have

$$\lim_{n\infty} x_n = \bar{x}.$$

- The equilibrium point $\bar{x}$ of Equation (2) is globally asymptotically stable if $\bar{x}$ is locally stable, and $\bar{x}$ is also a global attractor of Equation (2).
- The equilibrium point $\bar{x}$ of Equation (2) is unstable if $\bar{x}$ is not locally stable.

The linearized form of Equation (2) about the equilibrium $\bar{x}$ is the linear difference equation

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f(\bar{x}, \bar{x}, ..., \bar{x})}{\partial x_{n-i}} y_{n-i}.$$

Theorem 3. *Assume that $p, q \in \mathbb{R}$ and $k \in \{0, 1, 2, ...\}$. Then $|p| + |q| < 1$ is a sufficient condition for the asymptotic stability of the difference equation*

$$x_{n+1} + px_n + qx_{n-k} = 0, n = 0, 1, \dots$$

Remark 1. The theorem can be easily extended to a general linear equations of the form

$$x_{n+k} + p_1 x_{n+k-1} + ... + p_k x_n = 0, \quad n = 0, 1, \dots \tag{3}$$

where $p_1, p_2, ..., p_k \in \mathbb{R}$ and $k \geq 0$. Then Equation (3) is asymptotically stable provided that $\sum\limits_{i=0}^{k} |p_i| < 1$.

Consider the following equation

$$x_{n+1} = g(x_n, x_{n-3}, x_{n-4}). \tag{4}$$

The following theorem will be useful for the proof of our results in this paper.

Theorem 4. *Let $[a, b]$ be an interval of real numbers and assume that*

$$g : [a, b]^3 \quad [a, b],$$

is a continuous function satisfying the following properties:

(a) *$g(x, y, z)$ is nondecreasing in x and z in $[a, b]$ for each $y \in [a, b]$, and is nonincreasing in $y \in [a, b]$ for each x and z in $[a, b]$*
(b) *if $(m, M) \in [a, b] \times [a, b]$ is a solution of the system*

$$M = g(M, m, M), \qquad m = g(m, M, m),$$

then $m = M$.

Then (4) has a unique equilibrium point $\bar{x} \in [a, b]$ and every solution of (4) converges to $\bar{x}$.

2. Local stability of equation (1)

In this section we investigate the local stability character of the solutions of Equation (1). Equation (1) has a unique equilibrium point and is given by

$$\bar{x} = a\bar{x} + \frac{b\bar{x}^2}{c\bar{x} + d\bar{x}},$$

or

$$\bar{x}^2(1 - a)(c + d) = b\bar{x}^2,$$

then if $(1 - a)(c + d) \neq b$, then the unique equilibrium point is $\bar{x} = 0$.
Define the following function

$$f : (0, \infty)^3 \quad (0, \infty)$$
$$f(u, v, w) = au + \frac{buw}{cv + dw}.$$

It follows that

$$f_u(u, v, w) = a + \frac{bw}{cv + dw}, \quad f_v(u, v, w) = -\frac{bcuw}{(cv + dw)^2}, \quad f_w(u, v, w) = \frac{bcuv}{(cv + dw)^2}.$$

Then

$$f_u(\bar{x}, \bar{x}, \bar{x}) = a + \frac{b}{c + d}, \quad f_v(\bar{x}, \bar{x}, \bar{x}) = -\frac{bc}{(c + d)^2}, \quad f_w(\bar{x}, \bar{x}, \bar{x}) = \frac{bc}{(c + d)^2}.$$

The linearized equation of Equation (1) about $\bar{x}$ is

$$y_{n+1} - \left(a + \frac{b}{c+d}\right) y_n + \frac{bc}{(c+d)^2} y_{n-3} - \frac{bc}{(c+d)^2} y_{n-4} = 0. \tag{5}$$

Theorem 5. *Assume that*

$$b(d+3c) < (1-a)(c+d)^2.$$

Then the equilibrium point of Equation (1) is locally asymptotically stable.

Proof. It follows from Theorem 3 that Equation (5) is asymptotically stable if

$$\left| a + \frac{b}{c+d} \right| + \left| \frac{bc}{(c+d)^2} \right| + \left| \frac{bc}{(c+d)^2} \right| < 1,$$

or

$$a + \frac{b}{c+d} + \frac{2bc}{(c+d)^2} < 1,$$

and so,

$$\frac{b(d+3c)}{(c+d)^2} < (1-a).$$

The proof is complete. □

3. Global attractor of the equilibrium point of equation (1)

In this section we investigate the global attractivity character of solutions of Equation (1).

Theorem 6. *The equilibrium point $\bar{x}$ of Equation (1) is global attractor if $d(1-a) \neq b$*

Proof. Let p, q be real numbers and assume that $g : [p,q]^3 \quad [p,q]$ is a function defined by $g(u,v,w) = au + \frac{buw}{cv+dw}$, then we can easily see that the function $g(u,v,w)$ is increasing in u, w and decreasing in v. Suppose that (m, M) is a solution of the system

$$M = g(M, m, M), \qquad m = g(m, M, m).$$

Then from Equation (1), we see that

$$M = aM + \frac{bM^2}{cm + dM}, \qquad m = am + \frac{bm^2}{cM + dm}$$

or

$$M(1-a) = \frac{bM^2}{cm + dM}, \qquad m(1-a) = \frac{bm^2}{cM + dm}$$

then

$$c(1-a)mM + d(1-a)M^2 = bM^2, \qquad c(1-a)mM + d(1-a)m^2 = bm^2$$

subtracting, we obtain

$$d(1-a)(M^2 - m^2) = b(M^2 - m^2).$$

Since $d(1-a) \neq b$ therefore

$$M = m.$$

It follows from Theorem 4 that $\bar{x}$ is a global attractor of Equation (1), and then the proof is complete. □

4. Boundedness of solutions of equation (1)

In this section we study the boundedness of solutions of Equation (1).

Theorem 7. *Every solution of Equation (1) is bounded if $a + \frac{b}{d} < 1$.*

Proof. Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Equation (1). It follows from Equation (1) that

$$x_{n+1} = ax_n + \frac{bx_n x_{n-4}}{cx_{n-3} + dx_{n-4}} \leq ax_n + \frac{bx_n x_{n-4}}{dx_{n-4}} = \left(a + \frac{b}{d}\right) x_n.$$

Then $x_{n+1} \leq x_n, \quad \forall n \geq 0$. Then the sequence $\{x_n\}_{n=-4}^{\infty}$ is decreasing and so is bounded from above by $M = \max\{x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0\}$. □

For explaining the results of this section, we consider numerical example for $x_{-4} = 10, x_{-3} = 1, x_{-2} = 3, x_{-1} = 2, x_0 = 7$. (See Figure 1).

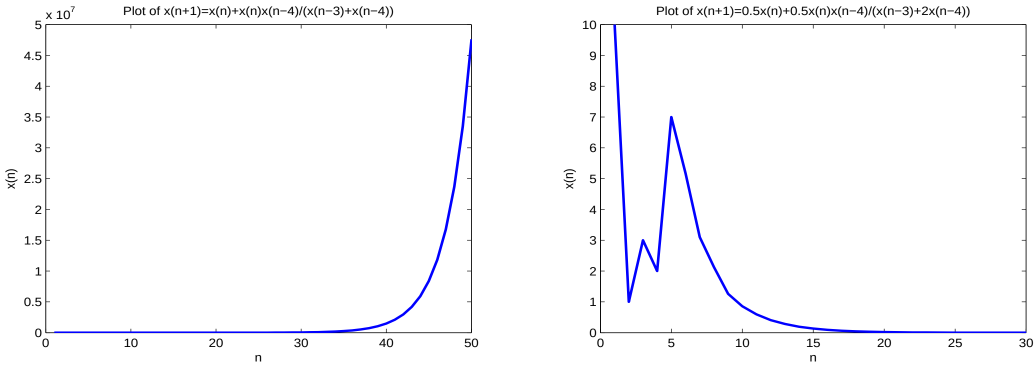

Figure 1. Left $a = 1, b = 1, c = 1, d = 1$ which don't satisfy the boundedness conditions (the solution is unbounded). Right $a = 0.5, b = 0.5, c = 1, d = 2$ which satisfy the boundedness conditions (the solution is bounded).

5. Special cases of equation (1)

Our goal in this section is to find a specific form of the solutions of some special cases of Equation (1) when , a, b, c and d are integers and give numerical examples of each case.

5.1. First case: on the difference equation $x_{n+1} = x_n + \frac{x_n x_{n-4}}{x_{n-3} + x_{n-4}}$

In this subsection we study the following special case of Equation (1):

$$x_{n+1} = x_n + \frac{x_n x_{n-4}}{x_{n-3} + x_{n-4}}, \quad n = 0, 1, \ldots \tag{6}$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzero real numbers.

Theorem 8. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Equation (6). Then for $n = 0, 1, 2, \ldots$*

$$\begin{cases} x_{4n} = r \prod_{i=1}^{n} \frac{(A_i f + 2B_i e)}{(B_i f + A_i e)} \frac{(A_i g + 2B_i f)}{(B_i g + A_i f)} \frac{(A_i h + 2B_i g)}{(B_i h + A_i g)} \frac{(A_i r + 2B_i h)}{(B_i r + A_i h)}, \\ x_{4n+1} = r \prod_{i=1}^{n+1} \frac{(A_i f + 2B_i e)}{(B_i f + A_i e)} \prod_{i=1}^{n} \frac{(A_i g + 2B_i f)}{(B_i g + A_i f)} \frac{(A_i h + 2B_i g)}{(B_i h + A_i g)} \frac{(A_i r + 2B_i h)}{(B_i r + A_i h)}, \\ x_{4n+2} = r \prod_{i=1}^{n+1} \frac{(A_i f + 2B_i e)}{(B_i f + A_i e)} \frac{(A_i g + 2B_i f)}{(B_i g + A_i f)} \prod_{i=1}^{n} \frac{(A_i h + 2B_i g)}{(B_i h + A_i g)} \frac{(A_i r + 2B_i h)}{(B_i r + A_i h)}, \\ x_{4n+3} = r \prod_{i=1}^{n+1} \frac{(A_i f + 2B_i e)}{(B_i f + A_i e)} \frac{(A_i g + 2B_i f)}{(B_i g + A_i f)} \frac{(A_i h + 2B_i g)}{(B_i h + A_i g)} \prod_{i=1}^{n} \frac{(A_i r + 2B_i h)}{(B_i r + A_i h)}, \end{cases}$$

where $x_{-4}=e, x_{-3}=f, x_{-2}=g, x_{-1}=h, x_0=r$, $\{A_m\}_{m=1}^{\infty}=\{1,3,7,17,41,...\}$, $\{B_m\}_{m=1}^{\infty}=\{1,2,5,12,29,...\}$ $A_m=2A_{m-1}+A_{m-2}, B_m=2B_{m-1}+B_{m-2}, m\geq 1, A_{-1}=-1,\ A_0=1,\ B_{-1}=1, B_0=0$, *or also* $A_m=2B_{m-1}+A_{m-1}, B_m=B_{m-1}+A_{m-1}, m\geq 0$, *and* $\prod_{i=1}^{0} G_i=1$.

Proof. For $n=0$, the result holds. Now suppose that our assumption holds for $n-1$ and for $n-2$. That is

$$\begin{cases} x_{4n-8}=r\prod_{i=1}^{n-2}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}, \\ x_{4n-7}=r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\prod_{i=1}^{n-2}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}, \\ x_{4n-6}=r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\prod_{i=1}^{n-2}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}, \\ x_{4n-5}=r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-2}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}, \end{cases}$$

$$\begin{cases} x_{4n-4}=r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}, \\ x_{4n-3}=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\prod_{i=1}^{n-1}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}, \\ x_{4n-2}=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\prod_{i=1}^{n-1}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}, \\ x_{4n-1}=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-1}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}. \end{cases}$$

Now it follows from Equation (6) that

$$\begin{aligned} x_{4n}&=x_{4n-1}+\frac{x_{4n-1}x_{4n-5}}{x_{4n-4}+x_{4n-5}}=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-1}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\Bigg(1+ \\ &\frac{r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-2}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}}{r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}+r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-2}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}}\Bigg) \\ &=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-1}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\left(1+\frac{1}{\frac{(A_{n-1}r+2B_{n-1}h)}{(B_{n-1}r+A_{n-1}h)}+1}\right) \\ &=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-1}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\left(1+\frac{B_{n-1}r+A_{n-1}h}{(A_{n-1}+B_{n-1})r+(2B_{n-1}+A_{n-1})h}\right) \\ &=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-1}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\ \frac{(A_{n-1}+2B_{n-1})r+(2B_{n-1}+2A_{n-1})h}{(A_{n-1}+B_{n-1})r+(2B_{n-1}+A_{n-1})h} \\ &=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\prod_{i=1}^{n-1}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\ \frac{A_nr+2B_nh}{B_nr+A_nh} \\ &=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}. \end{aligned}$$

Similarly,

$$x_{4n+1}=x_{4n}+\frac{x_{4n}x_{4n-4}}{x_{4n-3}+x_{4n-4}}=r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}$$

$$\left(1+\frac{r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}}{r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\prod_{i=1}^{n-1}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}+r\prod_{i=1}^{n-1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}}\right)$$

$$\begin{aligned}
&= r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\left(1+\frac{1}{\frac{(A_nr+2B_nh)}{(B_nr+A_nh)}+1}\right)\\
&= r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\left(1+\frac{B_nr+A_nh}{(A_n+B_{n-1})r+(2B_n+A_n)h}\right)\\
&= r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\frac{(A_n+2B_n)r+(2B_n+2A_n)h}{(A_n+B_n)r+(2B_n+A_n)h}\\
&= r\prod_{i=1}^{n}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}\frac{A_{n+1}r+2B_{n+1}h}{B_{n+1}r+A_{n+1}h}\\
&= r\prod_{i=1}^{n+1}\frac{(A_if+2B_ie)}{(B_if+A_ie)}\prod_{i=1}^{n}\frac{(A_ig+2B_if)}{(B_ig+A_if)}\frac{(A_ih+2B_ig)}{(B_ih+A_ig)}\frac{(A_ir+2B_ih)}{(B_ir+A_ih)}.
\end{aligned}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

For explaining the results of this section, we consider numerical example for $x_{-4}=10, x_{-3}=1, x_{-2}=3, x_{-1}=2, x_0=7$, (See Figure 2).

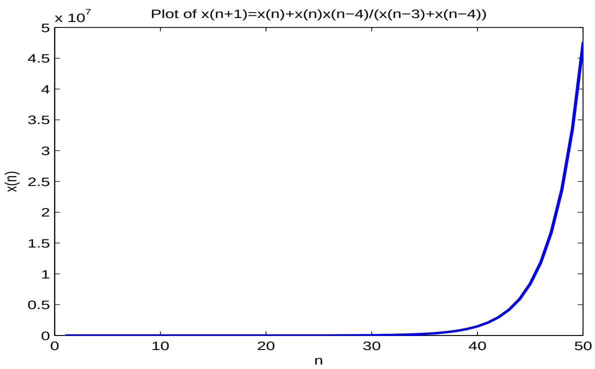

Figure 2. $x_{-4}=10, x_{-3}=1, x_{-2}=3, x_{-1}=2, x_0=7$

5.2. Second case: on the difference equation $x_{n+1}=x_n+\frac{x_nx_{n-4}}{x_{n-3}-x_{n-4}}$

In this subsection we study the following special case of Equation (1):

$$x_{n+1}=x_n+\frac{x_nx_{n-4}}{x_{n-3}-x_{n-4}}, \quad n=0,1,... \tag{7}$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzero real numbers.

Theorem 9. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Equation (7). Then for $n = 0, 1, 2, ...$*

$$\begin{cases} x_{8n-4} = \frac{r^n}{e^{n-1}} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\ x_{8n-3} = f\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\ x_{8n-2} = g\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\ x_{8n-1} = h\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\ x_{8n} = \frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\ x_{8n+1} = -f\frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^{n+1}(-g+f)^n(-h+g)^n(-r+h)^n}, \\ x_{8n+2} = fg\frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^{n+1}(-g+f)^{n+1}(-h+g)^n(-r+h)^n}, \\ x_{8n+3} = -fgh\frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^{n+1}(-g+f)^{n+1}(-h+g)^{n+1}(-r+h)^n}. \end{cases}$$

Proof. For $n = 0$, the result holds. Now suppose that our assumption holds for $n-1$. That is

$$\begin{cases} x_{8n-12} = \frac{r^{n-1}}{e^{n-2}} \frac{(rfgh)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-11} = f\frac{r^{n-1}}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-10} = g\frac{r^{n-1}}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-9} = h\frac{r^{n-1}}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-8} = \frac{r^n}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-7} = -f\frac{r^n}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^n(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-6} = fg\frac{r^n}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^n(-g+f)^n(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-5} = -fgh\frac{r^n}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^{n-1}}. \end{cases}$$

Now it follows from Equation (7) that

$$\begin{aligned} x_{8n} &= x_{8n-1} + \frac{x_{8n-1}x_{8n-5}}{x_{8n-4} - x_{8n-5}} = h\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \\ &\left(1 + \frac{-fgh\frac{r^n}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^{n-1}}}{\frac{r^n}{e^{n-1}} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} + fgh\frac{r^n}{e^{n-1}} \frac{(rfgh)^{n-1}}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^{n-1}}}\right) \\ &= h\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \left(1 + \frac{-1}{\frac{r}{(-r+h)} + 1}\right) \\ &= h\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \left(1 + \frac{r-h}{h}\right) \\ &= h\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \frac{r}{h} \\ &= \frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \end{aligned}$$

Similarly,

$$
\begin{aligned}
x_{8n+1} &= x_{8n} + \frac{x_{8n}x_{8n-4}}{x_{8n-3} - x_{8n-4}} \\
&= \frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \\
&\left(1 + \frac{\frac{r^n}{e^{n-1}} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}}{f\frac{r^n}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} - \frac{r^n}{e^{n-1}} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}}\right) \\
&= \frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \left(1 + \frac{1}{\frac{f}{e} - 1}\right) \\
&= \frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \left(1 + \frac{e}{f-e}\right) \\
&= \frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \frac{f}{f-e} \\
&= -f\frac{r^{n+1}}{e^n} \frac{(rfgh)^n}{(-f+e)^{n+1}(-g+f)^n(-h+g)^n(-r+h)^n}
\end{aligned}
$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Consider numerical examples which represent different types of solutions to Equation (7).
See Figure 3, since $x_{-4} = 20, x_{-3} = 10, x_{-2} = 30, x_{-1} = 2, x_0 = 10$. The solution is bounded and converges to $\bar{x} = 0$.
Now, if we take $x_{-4} = 1, x_{-3} = 3, x_{-2} = 1, x_{-1} = 40, x_0 = 10$, the solution is unbounded (see Figure 4).

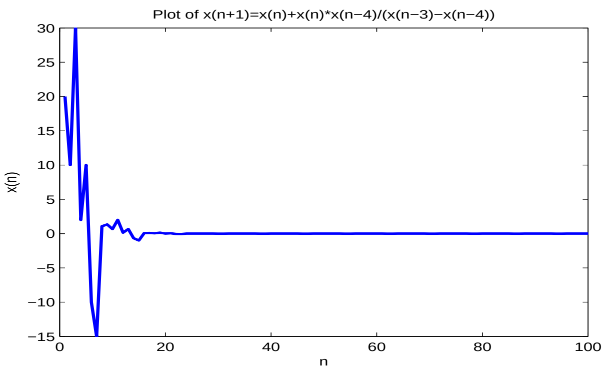

Figure 3. $x_{-4} = 20, x_{-3} = 10, x_{-2} = 30, x_{-1} = 2, x_0 = 10$. The solution is bounded and converges to $\bar{x} = 0$.

5.3. Third case: on the difference equation $x_{n+1} = x_n - \frac{x_n x_{n-4}}{x_{n-3} + x_{n-4}}$

In this subsection we study the following special case of Equation (1):

$$x_{n+1} = x_n - \frac{x_n x_{n-4}}{x_{n-3} + x_{n-4}}, \quad n = 0, 1, \ldots \tag{8}$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzero real numbers.

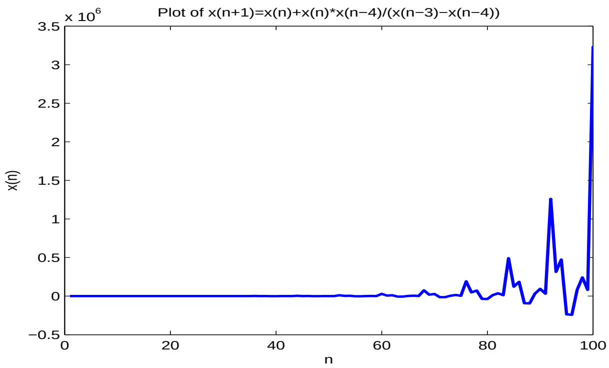

Figure 4. $x_{-4}=1, x_{-3}=3, x_{-2}=1, x_{-1}=40, x_0=10$, the solution is unbounded.

Theorem 10. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Equation (8). Then for $n=0,1,2,...$*

$$\begin{cases} x_{4n} &= \dfrac{r\,(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)(ir+h)}, \\ x_{4n+1} &= \dfrac{rf\,(rfgh)^n}{\prod\limits_{i=1}^{n+1}(if+e)\prod\limits_{i=1}^{n}(ig+f)(ih+g)(ir+h)}, \\ x_{4n+2} &= \dfrac{rfg\,(rfgh)^n}{\prod\limits_{i=1}^{n+1}(if+e)(ig+f)\prod\limits_{i=1}^{n}(ih+g)(ir+h)}, \\ x_{4n+3} &= \dfrac{(rfgh)^{n+1}}{\prod\limits_{i=1}^{n+1}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n}(ir+h)}. \end{cases}$$

Proof. For $n=0$, the result holds. Now suppose that our assumption holds for $n-1$ and for $n-2$. That is

$$\begin{cases} x_{4n-4} = \dfrac{r\,(rfgh)^{n-1}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)(ih+g)(ir+h)}, \\ x_{4n-3} = \dfrac{rf\,(rfgh)^{n-1}}{\prod\limits_{i=1}^{n}(if+e)\prod\limits_{i=1}^{n-1}(ig+f)(ih+g)(ir+h)}, \\ x_{4n-2} = \dfrac{rfg\,(rfgh)^{n-1}}{\prod\limits_{i=1}^{n}(if+e)(ig+f)\prod\limits_{i=1}^{n-1}(ih+g)(ir+h)}, \\ x_{4n-1} = \dfrac{(rfgh)^{n}}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-1}(ir+h)}, \end{cases}$$

$$\begin{cases} x_{4n-8} = \dfrac{r\,(rfgh)^{n-2}}{\prod\limits_{i=1}^{n-2}(if+e)(ig+f)(ih+g)(ir+h)}, \\ x_{4n-7} = \dfrac{rf\,(rfgh)^{n-2}}{\prod\limits_{i=1}^{n-1}(if+e)\prod\limits_{i=1}^{n-2}(ig+f)(ih+g)(ir+h)}, \\ x_{4n-6} = \dfrac{rfg\,(rfgh)^{n-2}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)\prod\limits_{i=1}^{n-2}(ih+g)(ir+h)}, \\ x_{4n-5} = \dfrac{(rfgh)^{n-1}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-2}(ir+h)}. \end{cases}$$

Now it follows from Equation (8) that

$$x_{4n} = x_{4n-1} - \frac{x_{4n-1}x_{4n-5}}{x_{4n-4}+x_{4n-5}} = \frac{(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-1}(ir+h)}$$

$$\left(1 - \frac{\frac{(rfgh)^{n-1}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-2}(ir+h)}}{\frac{r\,(rfgh)^{n-1}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)(ih+g)(ir+h)} + \frac{(rfgh)^{n-1}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-2}(ir+h)}}\right)$$

$$\begin{aligned}
&= \frac{(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-1}(ir+h)}\left(1-\frac{1}{\frac{r}{(n-1)r+h}+1}\right)\\
&= \frac{(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-1}(ir+h)}\left(1-\frac{(n-1)r+h}{nr+h}\right)\\
&= \frac{(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)\prod\limits_{i=1}^{n-1}(ir+h)}\left(\frac{r}{nr+h}\right)\\
&= \frac{r(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)(ir+h)}.
\end{aligned}$$

Similarly,

$$x_{4n+1} = x_{4n} - \frac{x_{4n}x_{4n-4}}{x_{4n-3}+x_{4n-4}} = \frac{r(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)(ir+h)}$$

$$\left(1 - \frac{\frac{r(rfgh)^{n-1}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)(ih+g)(ir+h)}}{\frac{rf\,(rfgh)^{n-1}}{\prod\limits_{i=1}^{n}(if+e)\prod\limits_{i=1}^{n-1}(ig+f)(ih+g)(ir+h)} + \frac{r(rfgh)^{n-1}}{\prod\limits_{i=1}^{n-1}(if+e)(ig+f)(ih+g)(ir+h)}}\right)$$

$$\begin{aligned}
&= \frac{r(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)(ir+h)}\left(1-\frac{1}{\frac{f}{nf+e}+1}\right)\\
&= \frac{r(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)(ir+h)}\left(1-\frac{nf+e}{(n+1)f+e}\right)\\
&= \frac{r(rfgh)^n}{\prod\limits_{i=1}^{n}(if+e)(ig+f)(ih+g)(ir+h)}\left(\frac{f}{(n+1)f+e}\right)\\
&= \frac{r(rfgh)^n}{\prod\limits_{i=1}^{n+1}(if+e)\prod\limits_{i=1}^{n}(ig+f)(ih+g)(ir+h)}.
\end{aligned}$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Consider numerical examples which represent different types of solutions to Equation (8).
Assume that $x_{-4} = 100, x_{-3} = 30, x_{-2} = 80, x_{-1} = 1, x_0 = 3$ (See Figure 5).
Now for $x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = 30, x_{-1} = 50, x_0 = 300$ (See Figure 6).

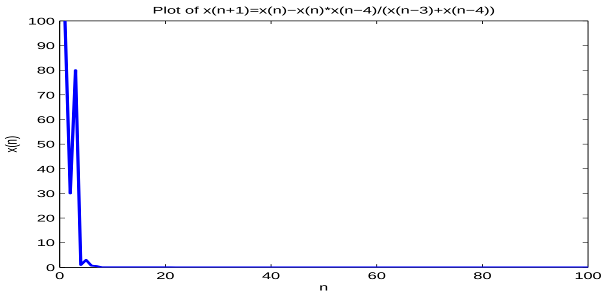

Figure 5. $x_{-4} = 100, x_{-3} = 30, x_{-2} = 80, x_{-1} = 1, x_0 = 3$.

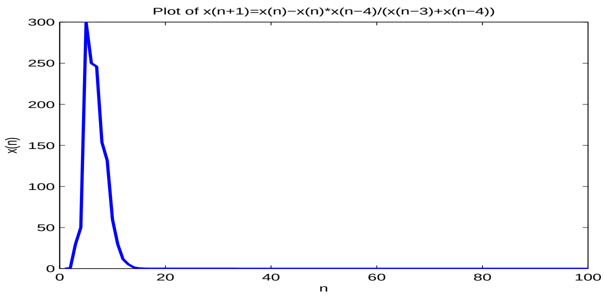

Figure 6. $x_{-4} = 0.1, x_{-3} = 0.5, x_{-2} = 30, x_{-1} = 50, x_0 = 300$.

5.4. Fourth case: on the difference equation $x_{n+1} = x_n + \dfrac{x_n x_{n-4}}{-x_{n-3} + x_{n-4}}$

In this subsection we study the following special case of Equation (1):

$$x_{n+1} = x_n + \frac{x_n x_{n-4}}{-x_{n-3} + x_{n-4}}, \quad n = 0, 1, ... \tag{9}$$

where the initial conditions $x_{-4}, x_{-3}, x_{-2}, x_{-1}, x_0$ are arbitrary nonzero real numbers.

Theorem 11. *Let $\{x_n\}_{n=-4}^{\infty}$ be a solution of Equation (9). Then for $n = 0, 1, 2, ...$*

$$\begin{cases}
x_{8n-4} = \dfrac{r^n}{e^{n-1}} \dfrac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\
x_{8n-3} = f\dfrac{r^n}{e^n} \dfrac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\
x_{8n-2} = g\dfrac{r^n}{e^n} \dfrac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\
x_{8n-1} = h\dfrac{r^n}{e^n} \dfrac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\
x_{8n} = \dfrac{r^{n+1}}{e^n} \dfrac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}, \\
x_{8n+1} = \dfrac{r^{n+1}}{e^n} \dfrac{(-f+2e)^{n+1}(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^{n+1}(-g+f)^n(-h+g)^n(-r+h)^n}, \\
x_{8n+2} = \dfrac{r^{n+1}}{e^n} \dfrac{(-f+2e)^{n+1}(-g+2f)^{n+1}(-h+2g)^n(-r+2h)^n}{(-f+e)^{n+1}(-g+f)^{n+1}(-h+g)^n(-r+h)^n}, \\
x_{8n+3} = \dfrac{r^{n+1}}{e^n} \dfrac{(-f+2e)^{n+1}(-g+2f)^{n+1}(-h+2g)^{n+1}(-r+2h)^n}{(-f+e)^{n+1}(-g+f)^{n+1}(-h+g)^{n+1}(-r+h)^n}.
\end{cases}$$

Proof. For $n = 0$, the result holds. Now suppose that our assumption holds for $n-1$. That is

$$\begin{cases} x_{8n-12} = \dfrac{r^{n-1}}{e^{n-2}} \dfrac{(-f+2e)^{n-1}(-g+2f)^{n-1}(-h+2g)^{n-1}(-r+2h)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-11} = f\dfrac{r^{n-1}}{e^{n-1}} \dfrac{(-f+2e)^{n-1}(-g+2f)^{n-1}(-h+2g)^{n-1}(-r+2h)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-10} = g\dfrac{r^{n-1}}{e^{n-1}} \dfrac{(-f+2e)^{n-1}(-g+2f)^{n-1}(-h+2g)^{n-1}(-r+2h)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-9} = h\dfrac{r^{n-1}}{e^{n-1}} \dfrac{(-f+2e)^{n-1}(-g+2f)^{n-1}(-h+2g)^{n-1}(-r+2h)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-8} = \dfrac{r^{n}}{e^{n-1}} \dfrac{(-f+2e)^{n-1}(-g+2f)^{n-1}(-h+2g)^{n-1}(-r+2h)^{n-1}}{(-f+e)^{n-1}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-7} = \dfrac{r^{n}}{e^{n-1}} \dfrac{(-f+2e)^{n}(-g+2f)^{n-1}(-h+2g)^{n-1}(-r+2h)^{n-1}}{(-f+e)^{n}(-g+f)^{n-1}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-6} = \dfrac{r^{n}}{e^{n-1}} \dfrac{(-f+2e)^{n}(-g+2f)^{n}(-h+2g)^{n-1}(-r+2h)^{n-1}}{(-f+e)^{n}(-g+f)^{n}(-h+g)^{n-1}(-r+h)^{n-1}}, \\ x_{8n-5} = \dfrac{r^{n}}{e^{n-1}} \dfrac{(-f+2e)^{n}(-g+2f)^{n}(-h+2g)^{n}(-r+2h)^{n-1}}{(-f+e)^{n}(-g+f)^{n}(-h+g)^{n}(-r+h)^{n-1}}. \end{cases}$$

Now it follows from Equation (9) that

$$\begin{aligned} x_{8n} &= x_{8n-1} + \frac{x_{8n-1}x_{8n-5}}{-x_{8n-4}+x_{8n-5}} \\ &= h\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n} \\ &\left(1+\frac{\frac{r^n}{e^{n-1}}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^{n-1}}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^{n-1}}}{-\frac{r^n}{e^{n-1}}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}+\frac{r^n}{e^{n-1}}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^{n-1}}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^{n-1}}}\right) \\ &= h\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{1}{-\frac{(-r+2h)}{(-r+h)}+1}\right) \\ &= h\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{(-r+h)}{-h}\right) \\ &= h\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(\frac{r}{h}\right) \\ &= \frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}. \end{aligned}$$

$$\begin{aligned} &= h\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{1}{-\frac{(-r+2h)}{(-r+h)}+1}\right) \\ &= h\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{(-r+h)}{-h}\right) \\ &= h\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(\frac{r}{h}\right) \\ &= \frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}. \end{aligned}$$

Similarly,

$$x_{8n+1} = x_{8n} + \frac{x_{8n}x_{8n-4}}{-x_{8n-3}+x_{8n-4}} = \frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}$$

$$\left(1+\frac{\frac{r^n}{e^{n-1}}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}}{-f\frac{r^n}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}+\frac{r^n}{e^{n-1}}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}}\right)$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{1}{-\frac{f}{e}+1}\right)$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{e}{-f+e}\right)$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(\frac{-f+2e}{-f+e}\right)$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^{n+1}(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^{n+1}(-g+f)^n(-h+g)^n(-r+h)^n}$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{1}{-\frac{f}{e}+1}\right)$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(1+\frac{e}{-f+e}\right)$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^n(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^n(-g+f)^n(-h+g)^n(-r+h)^n}\left(\frac{-f+2e}{-f+e}\right)$$
$$=\frac{r^{n+1}}{e^n}\frac{(-f+2e)^{n+1}(-g+2f)^n(-h+2g)^n(-r+2h)^n}{(-f+e)^{n+1}(-g+f)^n(-h+g)^n(-r+h)^n}.$$

Similarly, one can easily obtain the other relations. Thus, the proof is completed. □

Consider numerical examples which represent different types of solutions to Equation (9). The solution is unbounded since we choose $x_{-4}=20, x_{-3}=13, x_{-2}=3, x_{-1}=2, x_0=1$ (see Figure 7). However the solution converges to $\bar{x}=0$ by choosing $x_{-4}=100, x_{-3}=30, x_{-2}=10, x_{-1}=1, x_0=3$ (see Figure 8).

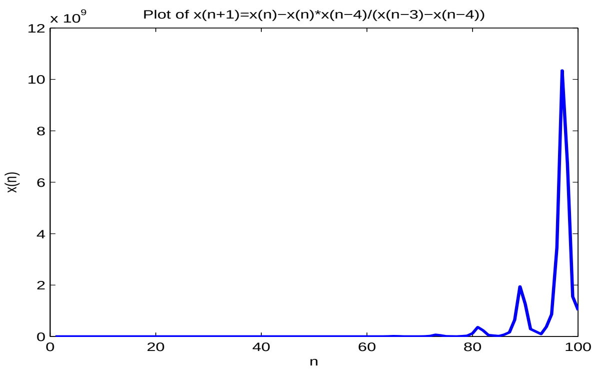

Figure 7. The solution is unbounded since we choose $x_{-4}=20, x_{-3}=13, x_{-2}=3, x_{-1}=2, x_0=1$.

6. Conclusion

This paper discussed global stability, boundedness, and the solutions of some special cases of Equation (1). In Section 2 we proved that if $b(d+3c)<(1-a)(c+d)^2$ then the equilibrium point of Equation (1) is locally asymptotically stable. In Section 3 we showed that the unique equilibrium of Equation (1) is globally asymptotically stable if $d(1-a)\neq b$. In Section 4 we proved that the solution of Equation (1) is bounded if $a+\frac{b}{d}<1$. In Section 5 we gave the form of the solution of four special cases of Equation (1) and gave numerical examples of each case.

Author Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

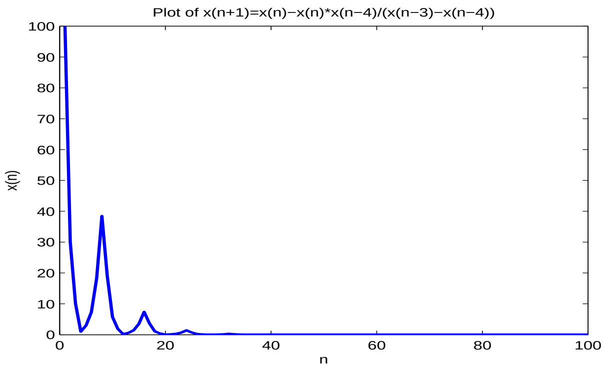

Figure 8. The solution converges to $\bar{x} = 0$ by choosing $x_{-4} = 100, x_{-3} = 30, x_{-2} = 10, x_{-1} = 1, x_0 = 3$

Conflicts of Interest: "The authors declare no conflict of interest."

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