On edge-prime cubic graphs with small components

Gee-Choon Lau[1],*, Sin-Min Lee[2] and Wai Chee Shiu[3,4]

[1]   Faculty of Computer & Mathematical Sciences, Universiti Teknologi MARA (Segamat Campus), 85000 Johor, Malaysia.

[2]   1304, North First Avenue, Upland, CA 91786 USA.; sinminlee@gmail.com

[3]   Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong.; shiu1958@gmail.com

[4]   College of Global Talents, Beijing Institute of Technology, Zhuhai, China.

*   Correspondence: geeclau@yahoo.com

Abstract: Let $G = G(V, E)$ be a (p, q)-graph. A bijection $f : E \to \{1, 2, 3, \ldots, q\}$ is called an edge-prime labeling if for each edge uv in E, we have $GCD(f^+(u), f^+(v)) = 1$ where $f^+(u) = \sum_{uw \in E} f(uw)$. A graph that admits an edge-prime labeling is called an edge-prime graph. In this paper we obtained some sufficient conditions for graphs with regular component(s) to admit or not admit an edge-prime labeling. Consequently, we proved that if G is a cubic graph with every component is of order $4, 6$ or 8, then G is edge-prime if and only if $G \not\cong K_4$ or $nK(3, 3)$, $n \equiv 2, 3 \pmod 4$. We conjectured that a connected cubic graph G is not edge-prime if and only if $G \cong K_4$.

Keywords: Prime labeling, edge-prime labeling, cubic graphs.

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1. Introduction

Let $G = (V(G), E(G))$ (or $G = (V, E)$ for short if not ambiguous) be a simple, finite and undirected (p, q)-graph of order $|V| = p$ and size $|E| = q$. For integers a, b with $a \le b$, let $[a, b] = \{n \in \mathbb{Z} \mid a \le n \le b\}$. All notation not defined in this paper can be found in [1].

The concept of prime labeling was originated by Entringer and it was introduced in a paper by Tout *et al.* [2]. A graph G with p vertices and q edges is said to have a prime labeling if function $f : V \to [1, p]$ is bijective and for every edge $e = uv$ of G, $GCD(f(u), f(v)) = 1$. For simplicity, we will use (a, b) to denote $GCD(a, b)$. Currently, the two most prominent open conjectures involving prime labelings are the following:

(1) All tree graphs have a prime labeling (Entringer-Tout Conjecture);
(2) All unicyclic graphs have a prime labeling (Seoud and Youssef [3]).

In 2011, Haxell and Pikhurko [4] proved that all large trees are prime. In 1991, Deretsky *et al.* [5] introduced the notion of dual of prime labeling which is known as vertex prime labeling. A graph with q edges has vertex prime labeling if its edges can be labeled with distinct integers $[1, q]$ such that for each vertex of degree at least 2, the greatest common divisor of the labels on its incident edges is 1. A conjecture: "Any 2-regular graph has a vertex prime labeling if and only if it does not have two odd cycles." was proposed.

An excellent survey on graph labeling is maintained by Gallian [6]. In [7], we introduce another prime labeling of graphs.

Definition 1. Let $G = (V, E)$ be a (p, q)-graph. A bijection $f : E \to [1, q]$ is called an *edge-prime labeling* if for each edge uv in E, we have $(f^+(u), f^+(v)) = 1$, where $f^+(u) = \sum_{uw \in E} f(uw)$. A graph that admits an edge-prime labeling is called an *edge-prime graph*.

Among others, we proved that all 2-regular graphs, complete bipartite graphs $K(2, n)$, the bipartite graph $K(2, n) + K(2, n)$ $(n \ge 2)$, disjoint union of paths with at most one P_2 component, the generalized theta graph having $n \ge 3$ internally disjoint paths of length 3 (respectively, 4), or having 3 internally disjoint paths of

length $n \geq 2$ and certain family of trees are edge-prime. It is a conjecture that all trees of diameter at least 3 are edge-prime.

Let $n(G)$ (or nG if no ambiguity) denote the disjoint union of n copies of graph G. In this paper we obtained some sufficient conditions for graphs with regular component(s) to admit or not admit an edge-prime labeling. Consequently, we proved that if G is a cubic graph with every component is of order $4, 6$ or 8, then G is edge-prime if and only if $G \ncong K_4$ or $nK(3,3)$, $n \equiv 2, 3 \pmod 4$. In what follows, we only consider cubic graphs unless specified otherwise.

It is clear that an edge labeling $f : E(G) \to [1, |E(G)|]$ such that for each edge uv, $f^+(u)$ and $f^+(v)$ are not both even, and $|f^+(u) - f^+(v)| = 2^m, m \geq 0$ is an edge-prime labeling.

Suppose f is an edge labeling of a graph $H = (V, E)$. For each edge xy of H, let $d_{xy} = |f^+(x) - f^+(y)|$. We shall use this notation throughout this paper.

1. We say f has Property (A) if $d_{xy} = 2^m$ for some $m \geq 0$, and each $xy \in E(H)$.
2. Let G be an edge-prime graph of size q. Suppose H is r-regular with an edge-prime labeling f. We say f has Property (B) if $(d_{xy}, rq) = d_{xy}$, for each $xy \in E(H)$. Moreover, f has Property (C) if for each $xy \in E(H)$, $d_{xy} = 2^m$ for some $m \geq 0$ or $(d_{xy}, rq) = d_{xy}$.

Theorem 2. *Let G be an edge-prime graph. Suppose H is an r-regular graph that admits an edge-prime labeling having Property (A).*

(a). If $r|E(G)|$ is even or $|E(G + H)|$ is odd, then $G + H$ is edge-prime.
(b). If $r|E(G)|$ is odd and $|E(H)|$ is even, then $G + H$ is edge-prime.

Proof. Let f_1 and f_2 be edge-prime labelings of G and H respectively with f_2 satisfying the given condition.
(a). Suppose $r|E(G)|$ is even. Define g such that $g(e) = f_1(e)$ for $e \in E(G)$, and $g(e) = f_2(e) + |E(G)|$ for $e \in E(H)$. Consider an edge $uv \in E(G + H)$. It suffices to consider $uv \in E(H)$. Without loss of generality, assume $f_2^+(u)$ is odd. Now, $g^+(u) = r|E(G)| + f_2^+(u)$ is odd.

$$(g^+(u), g^+(v)) = (r|E(G)| + f_1^+(u), r|E(G)| + f_2^+(v)) = (g^+(u), f_2^+(u) - f_2^+(v)) = (g^+(u), 2^m) = 1.$$

Suppose $|E(G + H)| = q$ is odd. Define g such that $g(e) = f_1(e)$ for $e \in E(G)$, and $g(e) = q + 1 - f_2(e)$ for $e \in E(H)$. Again, we only need to consider $uv \in E(H)$. Without loss of generality, assume $f_2^+(u)$ is odd. Now, $g^+(u) = r(q + 1) - f_2^+(u)$ is odd. We have

$$(g^+(u), g^+(v)) = (rq - f_2^+(u) + r, rq - f_2^+(v) + r) = (g^+(u), f_2^+(u) - f_2^+(v)) = (g^+(u), 2^m) = 1.$$

$$(g^+(u), g^+(v)) = (rq - f_2^+(u) + r, rq - f_2^+(v) + r) = (g^+(u), f_2^+(u) - f_2^+(v)) = (g^+(u), 2^m) = 1.$$

Hence, g is also an edge-prime labeling.
(b). Suppose $r|E(G)|$ is odd and $|E(H)|$ is even. Clearly, $|E(G)|$ is odd, and thus $|E(G + H)|$ is odd. From (a), we get that $G + H$ is also edge-prime. $\square$

Theorem 3. *Suppose G is a graph of even size q such that every component of G is regular. If G admits an edge-prime labeling f having Property (A), then nG is edge-prime for $n \geq 2$.*

Proof. For $k \geq 1$, define a partial edge labeling g_k for the k-th copy of G such that $g_k(e) = f(e) + (k - 1)q$. Clearly, every induced vertex label in G and the corresponding vertex in the k-th copy of nG have the same parity. Moreover, every 2 induced adjacent vertex labels differ by $2^m, m \geq 0$. Hence, nG is edge-prime. $\square$

Theorem 4. *Suppose every component of G is an even regular graph. If G admits an edge-prime labeling f having Property (A), then nG is edge-prime for $n \geq 2$.*

Proof. Let H be a component of G, which is r-regular. By Theorem 2 (a), since r is even, $G + H$ is edge-prime. We may repeat this procedure for each component of G to show that $2G$ is edge-prime and so is nG. $\square$

Lemma 5. *Suppose G is a graph of size q with an edge-prime labeling f. If H is an r-regular graph with an edge-prime labeling g having Property (B), then G + H is edge-prime.*

Proof. We define an edge labeling h for $G + H$ by

$$h(e) = \begin{cases} f(e), & \text{if } e \in E(G); \\ g(e) + q, & \text{if } e \in E(H). \end{cases}$$

We only need to consider edge $uv \in E(H)$. Now, $h^+(u) = g^+(u) + rq$ and $h^+(v) = g^+(v) + rq$. Since $(d_{uv}, g^+(v)) = (g^+(u), g^+(v)) = 1$ and rq is a multiple of d_{uv}, we have $(h^+(u), h^+(v)) = (g^+(u) + rq, g^+(v) + rq) = (d_{uv}, g^+(v) + rq) = 1$. Hence h is an edge-prime labeling and $G + H$ is edge-prime. $\qquad\square$

When rq is even, or rq is odd and $|E(H)|$ is even, we may relax the condition of Lemma 5 to have the following corollary.

Corollary 6. *Suppose G is a graph of size q with an edge-prime labeling f, and H is an r-regular graph with an edge-prime labeling g.*

(a). *Suppose rq is even. If g has Property (C), then G + H is edge-prime.*
(b). *Suppose rq is odd and $|E(H)|$ is even. If g has Property (C), then G + H is edge-prime.*

Proof. (a). We define an edge labeling h for $G + H$ as in Lemma 5. By the proof of Lemma 5, we only need to consider the case when $d_{uv} = 2^m$ with $m \geq 0$, where $uv \in E(H)$. Without loss of generality, we may assume $g^+(v)$ is odd. This implies that $h^+(v)$ is odd. By the same computation in the proof of Lemma 5, we have $(h^+(u), h^+(v)) = (d_{uv}, h^+(v)) = 1$. Hence $G + H$ is edge-prime.
(b). Suppose rq is odd and $|E(H)|$ is even. Clearly, q is odd and $|E(G + H)|$ is odd. Similar to (a), we only need to consider the case when $d_{uv} = 2^m$ with $m \geq 0$, where $uv \in E(H)$. By Theorem 2 (b), we have $G + H$ is edge-prime. $\qquad\square$

Theorem 7. *Suppose G is an r-regular graph of size q with an edge-prime labeling f. If f has Property (B), then nG is edge-prime.*

Proof. The theorem follows by applying Lemma 5 repeatedly. $\qquad\square$

Corollary 8. *Suppose G is an r-regular graph of size q with an edge-prime labeling f, where rq is even. If f has Property (C), then nG is edge-prime.*

Proof. This follows by applying Corollary 6 repeatedly. $\qquad\square$

Remark 1. Suppose G is an edge-prime graph of size q, where $q \equiv 0 \pmod 6$. Suppose H is a cubic graph with an edge-prime labeling g. If $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$ for all $xy \in E(H)$, then g has Property (C).

Remark 2. Note that the induced vertex labeling of each new edge labeling defined at each theorem or corollary above is difference preserved, i.e., all d_{xy} remain unchanged.

We next show the possible existence of non-edge-prime regular graphs.

Theorem 9. *Let G be a (p, q)-graph containing t component(s) such that every component of G is of size e_j with $e_j \equiv 1 \pmod 4$, $1 \leq j \leq t$. Let $f : E(G) \to [1, q]$ be any bijection. Suppose no 2 adjacent vertex labels of G under f^+ are even implies that every component of G receives odd number of odd edge labels under f. If $t \equiv 2$ or 3 $\pmod 4$, then G is not edge-prime.*

Proof. Let f be any bijective edge labeling of G. Suppose $t \equiv 2 \pmod 4$. Now $q \equiv 2 \pmod 4$. Hence, there are odd number of odd edge labels. There is a component receives even number of odd edge labels under f.

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By the hypothesis, there is a component containing two adjacent vertices whose labels are even. Hence f is not an edge prime labeling. That is, G does not admit an edge-prime labeling.

Similarly, if $t \equiv 3 \pmod 4$, then $q \equiv 3 \pmod 4$. Hence, there are even number of odd edge labels. There is a component receives even number of odd edge labels under f. Hence, G does not admit an edge prime labeling. $\qquad \square$

2. Cubic graphs with same order components

Lemma 10. *If h is an edge labeling of $G = (V, E)$, then there are even number of odd vertex labels.*

Proof. The lemma follows from

$$\sum_{u \in V} h^+(v) = 2 \sum_{e \in E} h(e). \tag{1}$$

$\qquad \square$

Corollary 11. *Suppose f is an edge labeling of a graph G such that no adjacent vertex labels are even.*

(1). *Suppose G contains a component $H = K_n$ for $n \geq 3$. If n is even, then all vertex labels in H are odd. If n is odd, then there is exactly one even vertex label in H.*

(2). *If G contains a component $K \cong K(m, n)$ with odd m and n, then K contains odd number of odd edge labels.*

Proof. (1). By the assumption, H admits at most one even vertex label. By Lemma 1, there is no even vertex label in H if n is even.

(2). Let (X, Y) be the bipartition of a component K. If K has even vertex labels, then the corresponding vertices lie in X, say. By Lemma 10, there are even number of even vertex labels. Each even vertex label incident with even number of odd edge label as well as each odd vertex label incident with odd number of odd edge label. So, K contains odd number of odd edge labels. $\qquad \square$

Theorem 12. *The graph nK_4 is edge-prime if and only if $n \geq 2$.*

Proof. (Necessity) We prove by contrapositive. Suppose there is an edge-prime labeling f of K_4. By Corollary 11, all 4 vertex labels are odd and distinct. Since each vertex label lies between 6 and 15. So the set of vertex labels is a subset of $\{7, 9, 11, 13, 15\}$ of size 4. Since the vertex labels are pairwise relatively prime and the sum of all 4 vertex labels is 42 (from (1)), there is no solution. Hence, K_4 is not edge-prime.

(Sufficiency) **(a).** Suppose $n = 2t$ with $t \geq 1$. The labeling of the two left-most graphs in Figure 1 shows that $G = 2K_4$ is edge-prime with $d_{xy} \in \{2, 4, 6, 8\}$.
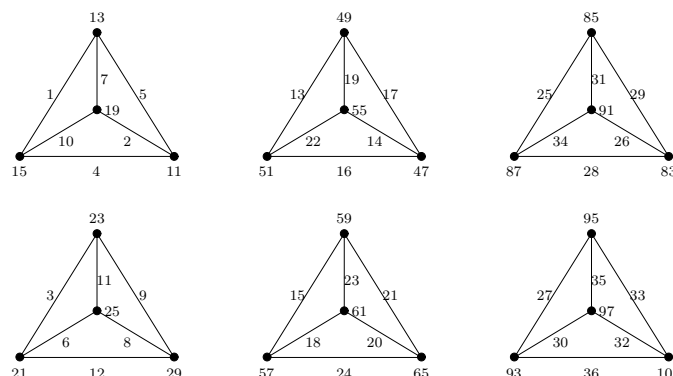

Figure 1. Edge-prime labeling of $6K_4$ with $d_{xy} \in \{2, 4, 6, 8\}$

By Remark 1 and Corollary 8, we conclude that $(2t)K_4$ is edge-prime for all $t \geq 1$.

(b). Suppose $n = 2t + 1$ with $t \geq 1$. If $t = 1$, then the labeling in Figure 2 shows that $H = 3K_4$ is edge-prime and $d_{xy} \in \{2, 4, 6, 8\}$.
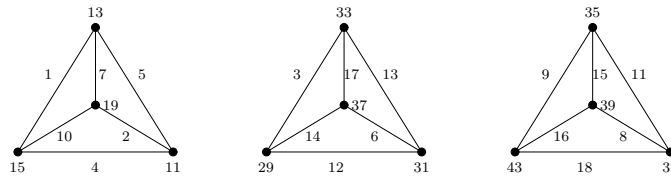
Figure 2. Edge-prime labeling of $3K_4$ with $d_{xy} \in \{2, 4, 6, 8\}$.

If $t \geq 2$, then we have just known that $G = (2t - 2)K_4$ is edge-prime. By Remark 1 and Corollary 6, $G + H = (2t + 1)K_4$ is edge-prime.   □

We now consider cubic graphs with every component of order 6. Each component must be a $C_3 \times K_2$ or a $K(3, 3)$.

Theorem 13. *For $n \geq 1$, $n(C_3 \times K_2)$ is edge-prime.*

Proof. In Figure 3, the labeling of the two top-left graphs shows that $H = C_3 \times K_2$ and $K = 2(C_3 \times K_2)$ are edge-prime with $d_{xy} \in \{2, 4, 6, 8\}$.
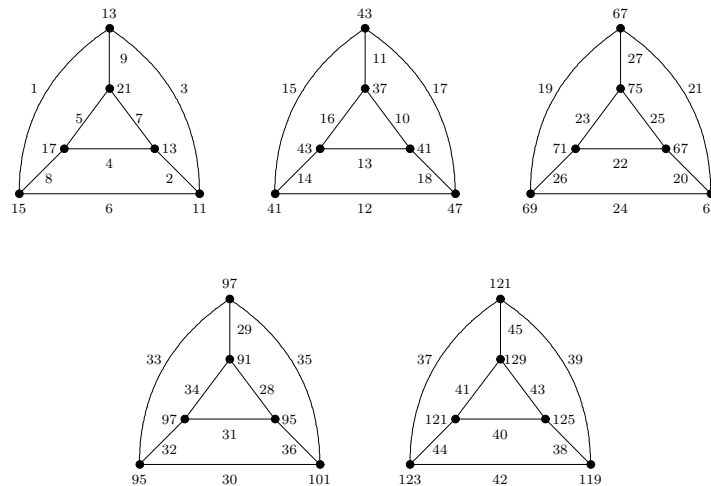
Figure 3. Edge-prime labeling of $5(C_3 \times K_2)$ with $d_{xy} \in \{2, 4, 6, 8\}$.

For $n = 2t \geq 2$, by Remark 1 and Corollary 8, we have $(2t)(C_3 \times K_2)$ is edge-prime for all $t \geq 1$.

Suppose $n = 2t + 1 \geq 1$. Since H is edge-prime, we only need to consider $2t + 1 \geq 3$. Since we have already known that $G = (2t)(C_3 \times K_2)$ is edge-prime, by Remark 1 and Corollary 6 $G + H = (2t + 1)(C_3 \times K_2)$ is edge-prime.   □

The next theorem shows that there are infinitely many non-edge-prime cubic graphs.

Theorem 14. *For $n \geq 1$, $n(K(3, 3))$ is edge-prime if and only if $n \equiv 0$ or $1 \pmod 4$.*

Proof. (Necessity) Suppose h is a bijective edge labeling of $n(K(3, 3))$ such that no 2 adjacent vertex labels are even. By Corollary 11 each component $K(3, 3)$ contains odd number of odd edge labels. So, $n(K(3, 3))$ satisfies the hypotheses of Theorem 9. Hence if $n(K(3, 3))$ is edge-prime, then $n \equiv 0$ or $1 \pmod 4$.
(Sufficiency) **(a).** Consider $n = 4t$. The labeling in Figure 4 shows that $G = 4K(3, 3)$ is edge-prime with $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$. Clearly, G satisfies the hypotheses of Corollary 8. Hence, we have $(4t)K(3, 3)$ is edge-prime for each $t \geq 1$.
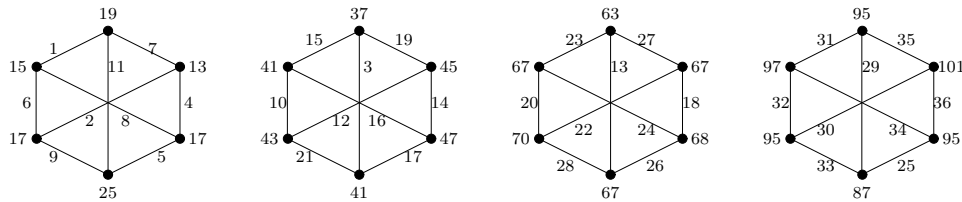
Figure 4. Edge-prime labeling of $4K(3,3)$ with $d_{xy} \in \{1,2,3,4,6,8\}$.

(b). Consider $n = 4t + 1$, $t \geq 0$. The labeling of the graph in Figure 5 shows that $H = K(3,3)$ is edge-prime with $d_{xy} \in \{4,8\}$.
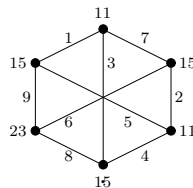

Figure 5. Edge-prime labeling of $K(3,3)$ with $d_{xy} \in \{4,8\}$

We assume that $t \geq 1$. Since $G = (4t)K(3,3)$ is edge-prime, by Remark 1 and Corollary 6 we have $G + H = (4t + 1)K(3,3)$ is edge-prime. $\qquad\square$

Theorem 15. *For $m, n \geq 1$, $m(C_3 \times K_2) + nK(3,3)$ is edge prime.*

Proof. Case 1. $n = 4t$, $t \geq 1$. From Theorem 14 we know that $(4t)K(3,3)$ is edge-prime (with $d_{xy} \in \{1,2,3,4,6,8\}$). From Theorem 13 we know that $m(C_3 \times K_2)$ admits an edge-prime labeling with $d_{xy} \in \{2,4,6,8\}$. Since the size of $(4t)K(3,3)$ is $36t$, by Remark 1 and Corollary 6 we have $(4t)K(3,3) + m(C_3 \times K_2)$ is edge-prime with $d_{xy} \in \{1,2,3,4,6,8\}$.
Case 2. $n = 4t + 1$, $t \geq 0$. Combining the labeling of $K(3,3)$ in Figure 5 and the labeling of $C_3 \times K_2$ in the top-middle of Figure 3 we have an edge-prime labeling of $(C_3 \times K_2) + K(3,3)$ (with $d_{xy} \in \{2,4,6,8\}$). From Theorem 13, Theorem 14 or Case 1, $(m-1)(C_3 \times K_2) + (4t)K(3,3)$ admits an edge-prime labeling g with $d_{xy} \in \{1,2,3,4,6,8\}$, $m \geq 1$ and $t \geq 0$. Since the size of $C_3 \times K_2$ is 18, g has Property (C). By Corollary 6 we obtain that $[(C_3 \times K_2) + K(3,3)] + [(m-1)(C_3 \times K_2) + (4t)K(3,3)] = m(C_3 \times K_2) + (4t + 1)K(3,3)$ is edge-prime.
Case 3. $n = 4t + 2$, $t \geq 0$. From Case 2 or Theorem 14, there is an edge-prime labeling of $(m-1)(C_3 \times K_2) + (4t + 1)K(3,3)$ with $d_{xy} \in \{1,2,3,4,6,8\}$ for $m \geq 1$ and $t \geq 0$. By the same argument as in Case 2, we have $[(C_3 \times K_2) + K(3,3)] + [(m-1)(C_3 \times K_2) + (4t + 1)K(3,3)] = m(C_3 \times K_2) + (4t + 2)K(3,3)$ is edge-prime.
Case 4. $n = 4t + 3$, $t \geq 0$. Consider the edge-prime labeling of $(C_3 \times K_2) + 3K(3,3)$ as shown in Figure 6.

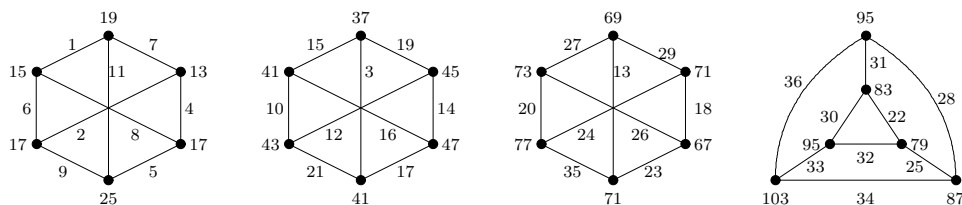**Figure 6.** Edge-prime labeling of $(C_3 \times K_2) + 3K(3,3)$ with $d_{xy} \in \{2,4,6,8,12,16\}$.

From Theorem 13, Theorem 14, or Case 1, $(m-1)(C_3 \times K_2) + (4t)K(3,3)$ admits an edge-prime labeling g with $d_{xy} \in \{1,2,3,4,6,8\}$, $m \geq 1$ and $t \geq 0$. Since the size of $(C_3 \times K_2) + 3K(3,3)$ is 36, by Corollary 6 we get that $m(C_3 \times K_2) + (4t + 3)K(3,3)$ are edge-prime. Note that the resulting labeling induces $d_{xy} \in \{1,2,3,4,6,8,12,16\}$. $\qquad\square$

It is known that there are exactly 5 connected cubic graphs of order 8. Figure 7 shows edge-prime labelings of these 5 graphs, denoted G_k, $1 \leq k \leq 5$.
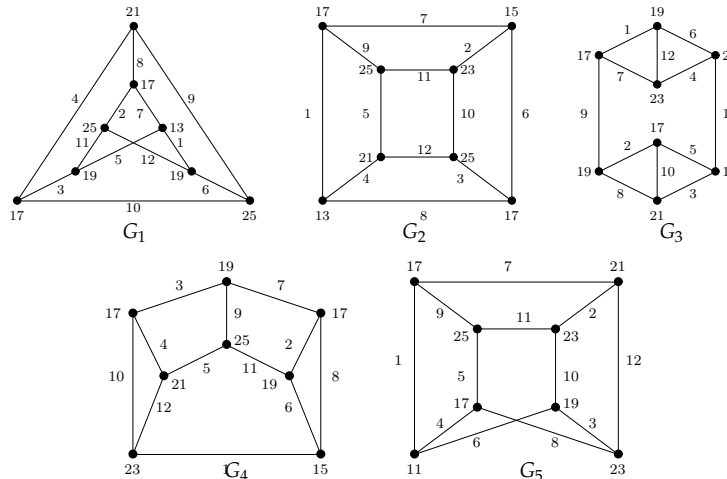
Figure 7. Edge-prime labelings for each cubic graph of order 8 with $d_{xy} \in \{2,4,6,8\}$.

Theorem 16. *If every component of G is a connected cubic graph of order 8, then G is edge-prime.*

Proof. Since each edge labeling shown in the Figure 7 induces $d_{xy} \in \{2,4,6,8\}$. By Remark 1 and applying Corollary 6 repeatedly, the theorem holds.   □

3. Cubic graphs with distinct order components

In this section, we completely determine the edge-primality of cubic graphs with components of distinct orders of 4, 6 or 8.

Theorem 17. *For $m, n \geq 1$, $mK_4 + n(C_3 \times K_2)$ is edge-prime.*

Proof. Suppose $m \geq 2$. From Table 1 we know that mK_4 and $n(C_3 \times K_2)$ are edge-prime when $m \geq 2$ and $n \geq 1$. Since $|E(mK_4)| = 6m$ and $d_{xy} \in \{2,4,6,8\}$ under the edge-prime labeling of $n(C_3 \times K_2)$, by Remark 1 and Corollary 6, $mK_4 + n(C_3 \times K_2)$ is edge-prime.

Suppose $m = 1$. Consider odd n. The edge-prime labeling of $K_4 + (C_3 \times K_2)$ shown in Figure 8 with $d_{xy} \in \{2,4,6,8\}$. For $n \geq 3$, from Table 1 we know that $(n-1)(C_3 \times K_2)$ is edge-prime with size $9(n-1)$. By Corollary 6, $(n-1)(C_3 \times K_2) + [K_4 + (C_3 \times K_2)] = K_4 + n(C_3 \times K_2)$ is edge-prime with $d_{xy} \in \{2,4,6,8\}$.
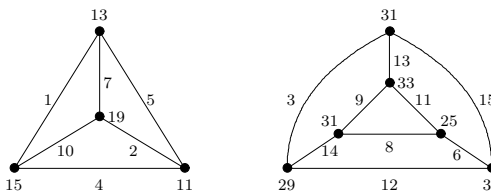

Figure 8. Edge-prime labeling of $K_4 + (C_3 \times K_2)$ with $d_{xy} \in \{2,4,6,8\}$.

Consider even n. Figure 9 shows that $K_4 + 2(C_3 \times K_2)$ is edge-prime. Using Table 1 and by Corollary 6, if needed, we get that $[K_4 + 2(C_3 \times K_2)] + (n-2)(C_3 \times K_2) = K_4 + n(C_3 \times K_2)$ is edge-prime with $d_{xy} \in \{2,4,6,8\}$.
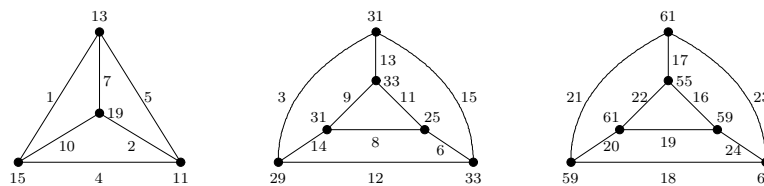

Figure 9. $K_4 + 2(C_3 \times K_2)$ is edge-prime with $d_{xy} \in \{2,4,6,8\}$.

     □

Theorem 18. *For $m, n \geq 1$, the graph $mK_4 + nK(3,3)$ is edge-prime.*

Proof. (a). Suppose $n \equiv 0, 1 \pmod 4$. Suppose m is even. By Table 1, we know that mK_4 and $nK(3,3)$ are edge-prime. Note that $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$ for $xy \in E(nK(3,3))$ and $|E(mK_4)| = 6m$. By Remark 1 and Corollary 6, we get the result.

Suppose m is odd. From the whole figure and the two top-left graphs of Figure 10 we see that $K_4 + K(3,3)$ and $K_4 + 4K(3,3)$ are edge-prime with $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$.
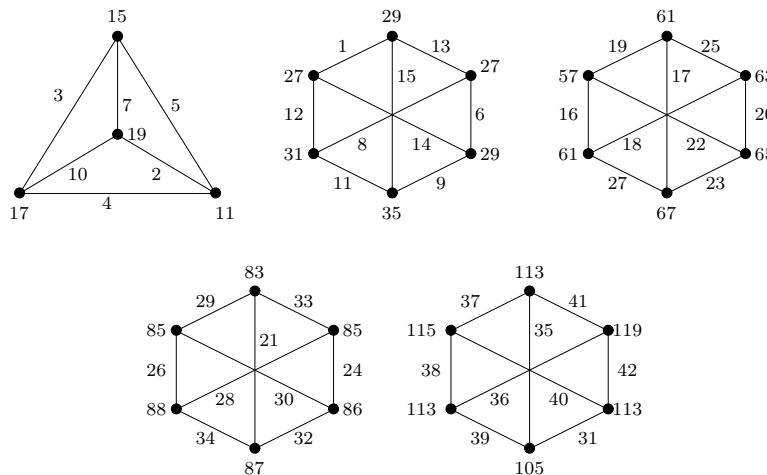

Figure 10. Edge-prime labeling of $K_4 + 4K(3,3)$ with $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$.

By Table 1 or the above case, there is an edge-prime labeling of $(m-1)K_4 + 4(t-1)K(3,3)$ for odd $m \geq 1$ and $t \geq 1$. Since $|E((m-1)K_4 + 4(t-1)K(3,3))| = 6(m-1) + 36(t-1)$, the labelings of $K_4 + K(3,3)$ and $K_4 + 4K(3,3)$ have Property (C). By Corollary 6 we get that $mK_4 + (4(t-1)+1)K(3,3)$ and $mK_4 + (4t)K(3,3)$ are edge-prime. Note that $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$ for the resulting edge labelings above.
(b). Suppose $n \equiv 2, 3 \pmod 4$. Figure 11 shows that $K_4 + 2K(3,3)$ is edge-prime.
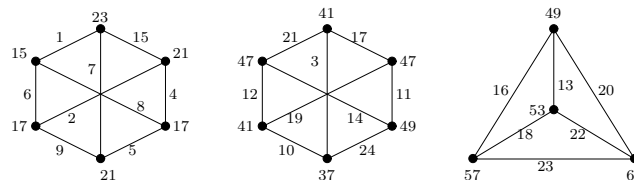

Figure 11. $K_4 + 2K(3,3)$ is edge-prime with $d_{xy} \in \{2, 4, 6, 8, 12, 16\}$.

From the above case, there is an edge-prime labeling g of $(m-1)K_4 + (n-2)K(3,3)$ with $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$ for $m \geq 1$ and $n \geq 2$. Since $|E(K_4 + 2K(3,3))| = 24$, g has Property (C). By Corollary 6 we have $mK_4 + nK(3,3)$ is edge-prime. $\square$

The following Table 1 gives a summary of the edge-prime labelings obtained above together with the set of the difference of adjacent vertex labels d_{xy}.

Table 1. Results from Theorems 12 to 18.

Graph G	Condition(s)	$\{d_{xy} \mid xy \in E(G)\}$
nK_4	$n \geq 2$	$\{2, 4, 6, 8\}$
$n(C_3 \times K_3)$	$n \geq 1$	$\{2, 4, 6, 8\}$
$nK(3,3)$	$n \equiv 0, 1 \pmod 4$	$\{1, 2, 3, 4, 6, 8\}$
$m(C_3 \times K_2) + nK(3,3)$	$m, n \geq 1$	$\{1, 2, 3, 4, 6, 8, 12, 16\}$
$mK_4 + n(C_3 \times K_2)$	$m, n \geq 1$	$\{2, 4, 6, 8\}$
$mK_4 + nK(3,3)$	$m, n \geq 1$	$\{1, 2, 3, 4, 6, 8, 12, 16\}$
$\sum_{i=1}^{5} n_i G_i$	$\sum_{i=1}^{5} n_i \geq 1$	$\{2, 4, 6, 8\}$

Theorem 19. *For $m_1, m_2, m_3 \geq 1$, the graph $m_1K_4 + m_2K(3,3) + m_3(C_3 \times K_2)$ is edge-prime.*

Proof. (a). Suppose m_2 is even. From Table 1, $m_1K_4 + m_2(K(3,3))$ is edge-prime and $m_3(C_3 \times K_2)$ admits an edge-prime labeling with $d_{xy} \in \{2,4,6,8\}$. Since the size of $m_1K_4 + m_2K(3,3)$ is $6(m_1 + 3m_2/2)$, by Corollary 6, we have the theorem.

(b). Suppose m_2 is odd. For even m_3, from the case (a) or Table 1, $m_1K_4 + (m_2 - 1)K(3,3) + m_3(C_3 \times K_2)$ is edge-prime with even size. From Figure 5 there is an edge-prime labeling of $K(3,3)$ with $d_{xy} \in \{4,8\}$. By Corollary 6 we have the theorem.

Now, assume that m_3 is odd. If $m_1 = m_2 = m_3 = 1$, then Figure 12 shows an edge-prime labeling of $K_4 + K(3,3) + (C_3 \times K_2)$.
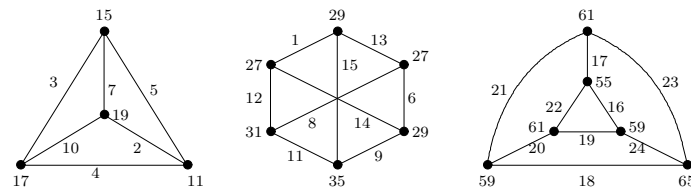

Figure 12. $K_4 + K(3,3) + (C_3 \times K_2)$ is edge-prime with $d_{xy} \in \{2,4,6,8\}$.

So we assume that at least one of m_1, m_2, m_3 is greater than 1. From Case (a) or Table 1, $m_1K_4 + (m_2 - 1)K(3,3) + (m_3 - 1)(C_3 \times K_2)$ is edge-prime with the size a multiple of 6. From the proof of Theorem 15 there is an edge-prime labeling of $(C_3 \times K_2) + K(3,3)$ with $d_{xy} \in \{2,4,6,8\}$. By Corollary 6 we have the theorem. $\square$

Remark 3. The set of d_{xy} of the edge-prime labeling obtained above is $\{1,2,3,4,6,8,12,16\}$.

Theorem 20. *The graph $mK_4 + \sum_{i=1}^5 n_iG_i$ is edge-prime for all $m \geq 1$ and $\sum_{i=1}^5 n_i \geq 1$.*

Proof. Suppose $m \geq 2$. From Table 1 and by Corollary 6 we get the theorem. Suppose $m = 1$. For $1 \leq k \leq 5$, we choose the smallest k such that $n_k > 0$ and label $K_4 + G_k$ from 1 to 18. The K_4 is labeled by $1,2,4,5,7,10$ as shown in Figure 9. The labeling of each G_k by $\{3,6,8,9\} \cup [11,18]$, if needed, is labeled by the remaining labels shown in Figure 13.
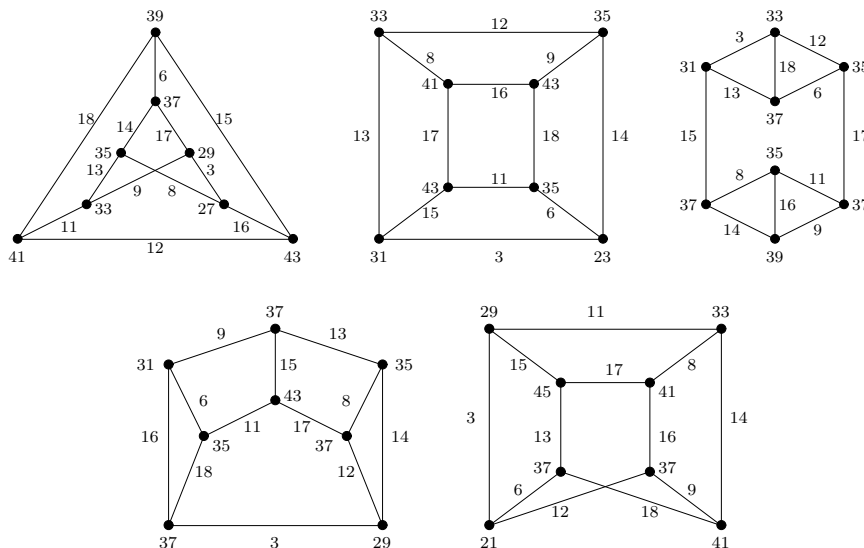

Figure 13. An edge labeling of G_1 to G_5 by $\{3,6,8,9\} \cup [11,18]$ with $d_{xy} \in \{2,4,6,8\}$; $\{2,8,12\}$; $\{2,4,6\}$; $\{2,4,6,8\}$; $\{4,8,16\}$.

The remaining unlabeled subgraph, if any, admits an edge-prime labeling with $d_{xy} \in \{2,4,6,8\}$ by Table 1. By Corollary 6 we obtain the theorem. $\square$

Theorem 21. *The graph $m(C_3 \times K_2) + \sum_{i=1}^5 n_iG_i$ is edge-prime for $m \geq 1$ and $\sum_{i=1}^5 n_i \geq 1$.*

Proof. From Table 1 we know that there are edge-prime labelings of $\sum_{k=1}^{5} n_i G_i$ and $m(C_3 \times K_2)$ with $d_{xy} \in \{2, 4, 6, 8\}$. Since the size of $\sum_{k=1}^{5} n_i G_i$ is 12 multiple, by Corollary 6 we have the theorem. □

Theorem 22. *The graph* $mK(3,3) + \sum_{i=1}^{5} n_i G_i$ *is edge-prime for* $m \geq 1$ *and* $\sum_{i=1}^{5} n_i \geq 1$.

Proof. (a). Suppose $m \equiv 0, 1 \pmod 4$. From Table 1 we know that there are edge-prime labelings of $\sum_{i=1}^{5} n_i G_i$ and $mK(3,3)$ with $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$. Since the size of $\sum_{i=1}^{5} n_i G_i$ is a 12 multiple, by Corollary 6 we have the theorem. Note that the set of difference adjacent vertex labels of the new edge-prime labeling is $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$.

Suppose $m \equiv 2, 3 \pmod 4$. For $1 \leq k \leq 5$, we choose the smallest k such that $n_k > 0$ and label $2K(3,3) + G_k$ from 1 to 30. Figure 4 shows an edge labeling of $2K(3,3)$ labeled by integers in $[1, 21] \setminus \{13, 18, 20\}$ with $d_{xy} \in \{2, 4, 6, 8\}$. The labeling of each G_k by $\{13, 18, 20\} \cup [22, 30]$, if needed, is shown in Figure 14.
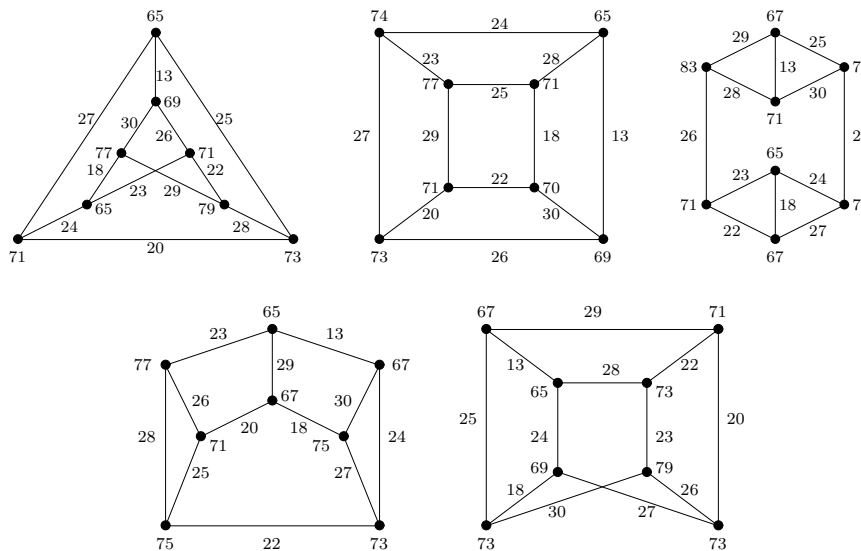

Figure 14. An edge labeling of graphs G_1 to G_5 by $\{13, 18, 20\} \cup [22, 30]$ with $d_{xy} \in \{2, 4, 6, 8, 12\}$; $\{1, 2, 3, 4, 6, 9\}$; $\{2, 4, 6, 8, 12, 16\}$; $\{2, 4, 6, 8, 12\}$; $\{2, 4, 6, 8\}$.

From Case (a) or Table 1 there is an edge-prime labeling of $(m-2)K(3,3) + \sum_{\substack{1 \leq i \leq 5 \\ i \neq k}} n_i G_i$, if any, with $d_{xy} \in \{1, 2, 3, 4, 6, 8\}$. Since the size of $2K(3,3) + G_k$ is 30, by Corollary 6 we have the theorem. □

Theorem 23. *For* $m_1, m_2 \geq 1$ *and* $\sum_{i=1}^{5} n_i \geq 1$, *the graph* $m_1 K_4 + m_2(C_3 \times K_2) + \sum_{i=1}^{5} n_i G_i$ *is edge-prime.*

Proof. The size of $\sum_{i=1}^{5} n_i G_i$ is a multiple of 12. From Table 1 and by Corollary 6 we have the theorem. □

Theorem 24. *For* $m_1, m_2 \geq 1$ *and* $\sum_{i=1}^{5} n_i \geq 1$, *the graph* $m_1 K_4 + m_2 K(3,3) + \sum_{i=1}^{5} n_i G_i$ *is edge-prime.*

Proof. The size of $\sum_{i=1}^{5} n_i G_i$ is a multiple of 12. From Table 1 and by Corollary 6 we have the theorem. □

Theorem 25. *For* $m_1, m_2 \geq 1$ *and* $\sum_{i=1}^{5} n_i \geq 1$, *the graph* $m_1(C_3 \times K_2) + m_2 K(3,3) + \sum_{i=1}^{5} n_i G_i$ *is edge-prime.*

Proof. The size of $\sum_{i=1}^{5} n_i G_i$ is a multiple of 12. From Table 1 and by Corollary 6 we have the theorem. □

Theorem 26. *Let* $m_1, m_2, m_3, \sum_{i=1}^{5} n_i \geq 1$, *the graph* $m_1 K_4 + m_2(C_3 \times K_2) + m_3 K(3,3) + \sum_{i=1}^{5} G_i$ *is edge-prime.*

Proof. The size of $\sum_{i=1}^{5} n_i G_i$ is a multiple of 12. From Remark 3 and by Corollary 6 we have the theorem. □

Corollary 27. *If G is a cubic graph with every component of order 4, 6 or 8, then G is edge-prime if and only if* $G \not\cong K_4$ *or* $nK(3,3)$, $n \equiv 2, 3 \pmod 4$.

In [7], we proved that (i) a 1-regular graph is edge-prime if and only if it is K_2, (ii) all 2-regular graphs are edge-prime, and (iii) if G is edge-prime, then $G + C_n$ ($n \geq 3$) and $G + K(1,2)$ are edge-prime.

Corollary 28. *Let G be an edge-prime graph as in Sections 2 and 3. For $\sum m_k \geq 1, n_k \geq 3$, $G + \sum m_k C_{n_k}$ and $G + K(1,2)$ are edge-prime.*

4. Open problems

Problem 29. Determine the edge-primality of the following families of cubic graphs.

(a) cylinder graph $C_n \times K_2$, $n \geq 5$.
(b) Mobiüs ladder $M(2n)$, $n \geq 2$.
(c) generalized Petersen graph $P(n,k)$, $n \geq 5, k \geq 2$.

It is easy to verify that K_n is edge-prime for $n = 2, 3, 5, 6, 7$ but not $n = 4$, and $K(n,n)$ is edge-prime for $n = 2, 3, 4$. We end with the following conjecture.

Conjecture 4.1. *For $n \geq 2$, we have*

(a) *K_n is edge-prime if and only if $n \neq 4$.*
(b) *$K(n,n)$ is edge-prime.*
(c) *All connected cubic graphs, except K_4, are edge-prime.*

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