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Banach Space X-Valued Bilateral Sequence Space $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$

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Abstract

Aims/ objectives:
In this paper we introduce and study vector–valued bilateral sequence space $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. We investigate the conditions connected with the comparison of the classes in terms of $\bar{\lambda}$ and $\bar{p}$ so that a class is contained in or equal to another class of same kind. We also study topological linear structure of this space when this space is topolized by a suitable paranorm.

Keywords: Bilateral Sequence, Sequence Space and Paranormed space
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1 Introduction

So far, a good number of research works have been done on bilateral sequence spaces for instance, see [2], [7], [3], [12] and [13]. Leon and Montes, in [9] has worked on the complex bilateral sequence space $\ell^2(\mathbb{Z})$ to obtain various results on hypercyclic bilateral weighted shift. In [8], Menet generalized this result to the complex bilateral sequence spaces $\ell^p(\mathbb{Z})$ with $1 \leq p < \infty$ and $c_\circ(\mathbb{Z})$ and afterwards, to the complex weighted spaces $\ell^p(v, \mathbb{Z})$ and $c_\circ(v, \mathbb{Z})$. Shkarin, in [11] and [10] used the bilateral sequence spaces $\ell_\infty(\mathbb{Z})$, $\ell_p(\mathbb{Z})$ with $1 \leq p < \infty$ and $c_\circ(\mathbb{Z})$ to obtain various results associated with weighted bilateral shift on these spaces and also used $\{f_j\}_{j\in\mathbb{Z}}$, a sequence of elements of $\mathcal{B}$ where $\mathcal{B}$ is a Banach space. We have introduced and studied the Banach space $X-$valued bilateral sequence spaces $c_\circ(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, $c(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ in [15] involving bilateral sequence $\bar{p} = (p_k)_{-\infty}^{\infty}$ and the multiplier $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$.

By a bilateral sequence we mean a function whose domain is the set $\mathbb{Z}$ of all integers with natural ordering. We will denote a bilateral sequence by the symbol $(a_k)_{-\infty}^{\infty}$ or $\bar{a} = (a_k)_{-\infty}^{\infty}$.As usual by the convergence of the bilateral series $\sum_{-\infty}^{\infty} a_k$ to s written as $\sum_{-\infty}^{\infty} a_k = s$ we shall mean the convergence

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of the sequence $(s_n)_{n=1}^{\infty}$ to s. Let $\bar{p} = (p_k)_{-\infty}^{\infty}$ and $\bar{q} = (q_k)_{-\infty}^{\infty}$ be bilateral sequences of strictly positive real numbers and $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$ and $\bar{\mu} = (\mu_k)_{-\infty}^{\infty}$ be bilateral sequences of non-zero complex numbers. Our aim in this paper is to investigate the results concerning the class $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ defined below as a generalization of the sequence space $\ell(X, \bar{\lambda}, \bar{p})$ (studied by Srivastava and Srivastava [16]) which is itself generalization of well known complex sequence space $\ell(\bar{p})$, $\bar{p} = (p_k)_1^{\infty}$ studied by Maddox [6] and many others, for instance, see [13], [**?**], [**?**] and [1]. We define

$$\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \left\{ \bar{x} = (x_k)_{-\infty}^{\infty} : x_k \in X,\ k \in \mathbb{Z},\ \left(\sum_{-\infty}^{\infty} ||\lambda_k x_k||^{p_k} \right) < \infty \right\}.$$

If $p_k = 1$ for all $k \in \mathbb{Z}$ in $\bar{p} = (p_k)_{-\infty}^{\infty}$, we shall denote $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ by $\ell(\mathbb{Z}, X, \bar{\lambda})$ and if $\lambda_k = 1$ for all $k \in \mathbb{Z}$ in $\bar{\lambda} = (\lambda_k)_{-\infty}^{\infty}$ then we shall denote $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ by $\ell(\mathbb{Z}, X, \bar{p})$.

We shall also need

$$\ell_\infty(\mathbb{Z}, \mathbb{R}) = \{\bar{a} = (a_k)_{-\infty}^{\infty} : a_k \in \mathbb{R},\ k \in \mathbb{Z},\ \text{sup}_k |a_k| < \infty\}$$

Throughout the paper, for each $k \in \mathbb{Z}$, we shall denote $t_k = \left|\frac{\lambda_k}{\mu_k}\right|^{p_k}$ and $M =$max$(1,$sup$_k p_k)$ and shall denote by $\mathbb{Z}(m, n)$ the open integer interval defined as

$$z(m,n) = \begin{cases} m+1, m+2, \ldots, n-2, n-1, & m+1 \leq n-1 \\ \phi, & \text{otherwise} \end{cases}$$

Also we shall denote complement of $\mathbb{Z}(m, n)$ by $\mathbb{Z} \backslash \mathbb{Z}(m, n)$.

Definition 1.1. Let X be a linear space. A mapping $g : X \to \mathbb{R}$ is called a paranorm if it satisfies following conditions :

(i) $g(\theta) = 0$

(ii) $g(x) = g(-x)$

(iii) $g(x + y) \leq g(x) + g(y)$

(iv) if (α_n) is a sequence of scalars with $\alpha_n \to \alpha$ and (x_n) is a sequence in X with $g(x_n - x) \to 0$ then $g(\alpha_n x_n - \alpha x) \to 0$ (continuity of scalar multiplication).

The paranorm is called total if

(v) $g(x) = 0$ implies $x = 0$, see [17].

2 Containment

In this section conditions for containment relations of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ in terms of $\bar{p}$ and $\bar{\lambda}$ are investigated.

Lemma 2.1. $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ *if and only if*

$$\lim_{k \to -\infty} \inf\ t_k > 0 \ \textit{and} \ \lim_{k \to \infty} \inf\ t_k > 0$$

Proof. Suppose $\lim_{k \to -\infty} \inf\ t_k > 0$ and $\lim_{k \to \infty} \inf\ t_k > 0$ and $\bar{x} = (x_k)_{-\infty}^{\infty} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Then there exists $m > 0$ such that $m|\mu_k|^{p_k} < |\lambda_k|^{p_k}$, for all sufficiently large values of $|k|$. Thus $m||\mu_k x_k||^{p_k} < ||\lambda_k x_k||^{p_k}$, for all sufficiently large values of $|k|$. Now we easily get that$\bar{x} \in \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$. Hence $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$.

Conversely suppose that $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ but $\lim_{k\to-\infty} \inf t_k = 0$ and/or $\lim_{k\to\infty} \inf t_k = 0$. Let us take the case when only $\lim_{k\to\infty} \inf t_k = 0$. Now we can find a sequence of integers $(k(n))$ such that $1 \le k(n) < k(n+1)$, $n \ge 1$ for which $n|\lambda_{k(n)}|^{p_{k(n)}} < |\mu_{k(n)}|^{p_{k(n)}}$. We now see that $\bar{x} = (x_k)_{-\infty}^{\infty}$ defined by

$$x_k = \begin{cases} \lambda_{k(n)}^{-1} n^{-2/p_{k(n)}} z, & \text{if } k = k(n), n \ge 1 \text{ and} \\ \theta, & \text{otherwise} \end{cases}$$

where $z \in X$, and $||z|| = 1$, is in $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ but not in $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ as

$$\sum_{k=-\infty}^{\infty} ||\lambda_k x_k||^{p_k} < \infty \text{ and}$$

$$\sum_{k=-\infty}^{\infty} ||\mu_k x_k||^{p_k} = \sum_{n=1}^{\infty} \left|\frac{\mu_{k(n)}}{\lambda_{k(n)}}\right|^{p_{k(n)}} . \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n^2} . n = \infty,$$

a contradiction to our assumption.

Similar proof can be given for the other cases. This completes the proof. □

Lemma 2.2. $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ *if and only if*

$$\lim_{k\to-\infty} \sup t_k < \infty \quad \textit{and} \quad \lim_{k\to\infty} \sup t_k < \infty$$

Proof. Sufficiency of the condition can easily be proved on the lines of above **Lemma 2.1**. For the necessity let us suppose that $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ but $\lim_{k\to-\infty} \sup t_k = \infty$ and/or $\lim_{k\to\infty} \sup t_k = \infty$. We take the case when only $\lim_{k\to\infty} \sup t_k = \infty$.Then there exits a sequence of integers $(k(n))$ such that, $1 \le k(n) < k(n+1)$, $n \ge 1$ for which

$$|\lambda_{k(n)}|^{p_{k(n)}} > n|\mu_{k(n)}|^{p_{k(n)}}.$$

Let $z \in X$ with $||z|| = 1$ and consider the sequence $\bar{x} = (x_k)_{-\infty}^{\infty}$ defined in the proof of Lemma 2.1. We easily see that $\bar{x} = (x_k)_{-\infty}^{\infty}$ is in $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ but $\bar{x} \notin \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ as

$$\sum_{k=-\infty}^{\infty} ||\lambda_k x_k||^{p_k} = \sum_{n=1}^{\infty} \left|\frac{\lambda_{k(n)}}{\mu_{k(n)}}\right|^{p_{k(n)}} . \frac{1}{n^2} > \sum_{n=1}^{\infty} \frac{1}{n^2} . n = \infty,$$

a contradiction to our assumption that $\ell(\mathbb{Z}, X, \bar{\mu}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

This completes the proof. □

When Lemmas 2.1 and 2.2 are combined, we get:

Theorem 2.3. $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \ell(\mathbb{Z}, X, \bar{\mu}, \bar{p})$ *if and only if*

$$0 < \lim_{k\to-\infty} \inf t_k < \lim_{k\to-\infty} \sup t_k < \infty \textit{ and}$$
$$0 < \lim_{k\to\infty} \inf t_k < \lim_{k\to\infty} \sup t_k < \infty.$$

Corollary 2.4. ***(i)*** $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{p})$ *if and only if*
$\lim_{k\to-\infty} \inf |\lambda_k|^{p_k} > 0$ *and* $\lim_{k\to\infty} \inf |\lambda_k|^{p_k} > 0$.
(ii) $\ell(\mathbb{Z}, X, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ *if and only if*
$\lim_{k\to-\infty} \sup |\lambda_k|^{p_k} < \infty$ *and* $\lim_{k\to\infty} \sup |\lambda_k|^{p_k} < \infty$.
(iii) $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) = \ell(\mathbb{Z}, X, \bar{p})$ *if and only if*
$0 < \lim_{k\to-\infty} \inf |\lambda_k|^{p_k} < \lim_{k\to-\infty} \sup |\lambda_k|^{p_k} < \infty$ *and*
$0 < \lim_{k\to\infty} \inf |\lambda_k|^{p_k} < \lim_{k\to\infty} \sup |\lambda_k|^{p_k} < \infty$.

Proof. Proof easily follows from Lemma 2.1, Lemma 2.2 and Theorem 2.3.

□

Lemma 2.5. *If* $p_k \le q_k$ *for all but finitely many* $k \in \mathbb{Z}$ *then*

$$\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{q})$$

Proof. Let $p_k \le q_k$ for all but finitely many $k \in \mathbb{Z}$. If $\bar{x} = (x_k)_{-\infty}^{\infty} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ then clearly $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{q})$ because $||\lambda_k x_k|| \le 1$ for all large values of $|k|$ and so $||\lambda_k x_k||^{q_k} \le ||\lambda_k x_k||^{p_k}$, for all large values of $|k|$. This completes the proof.

□

Theorem 2.6. *If*
(i) $\lim_{k\to-\infty} \inf\ t_k > 0$ *and* $\lim_{k\to\infty} \inf\ t_k > 0$*, and*
(ii) $p_k \le q_k$*, for all but finitely many* $k \in \mathbb{Z}$*, then*

$$\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \subset \ell(\mathbb{Z}, X, \bar{\mu}, \bar{q})$$

Proof. Proof easily follows from Lemmas 2.1 and 2.5.

□

3 Paranormed Space Structure of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$

As usual for the sequence spaces, here also the linear space structure of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ *over the field* $\mathbb{C}$ *of complex numbers is concerned, vector operations will be taken co-ordinatewise i.e.,* $\bar{x} + \bar{y} = (x_k + y_k)_{-\infty}^{\infty}$ *and* $\alpha\bar{x} = (\alpha x_k)_{-\infty}^{\infty}$*. Further we note that* $\bar{p} = (p_k)_{-\infty}^{\infty} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$ *is a necessary and sufficient condition for the linearity of* $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$*. Therefore throughout the section we shall take* $\bar{p} = (p_k)_{-\infty}^{\infty} \in \ell_\infty(\mathbb{Z}, \mathbb{R})$*.*

We define

(3.1) $$Q_{\bar{\lambda},\bar{p}}(\bar{x}) = \left(\sum_{-\infty}^{\infty} ||\lambda_k x_k||^{p_k} \right)^{1/M}$$

for $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ *where* $M = max(1, sup_k p_k)$*.*

recall the following definition (see [15]):

Definition 3.1. Let $E(X)$ be the linear space of normed space $X-$valued bilateral sequences $\bar{x} = (x_k)_{-\infty}^{\infty}$ and $x \in X$. We define
(i) $\delta_n(x) = (\dots, \theta, x, \theta, \dots)$, where x is at nth place, $n \in \mathbb{Z}$.
(ii) $E(X)$ equipped with the linear topology $\mathcal{T}$ is said to be a $GK-$space if the map $P_n : E(X) \to X, P_n(\bar{x}) = x_n$, is continuous for each $n \in \mathbb{Z}$.

A $GK-$space is called
(iii) a $GAD-$space if $\Phi(X)$ is dense in $E(X)$, where
$\Phi(X) = \{\bar{x} = (x_k)_{-\infty}^{\infty} : x_k \in X, k \in \mathbb{Z}\ and\ x_k = \theta,\ for\ all\ but\ finitely\ many\ k\}$,
(iv) a $GAK-$space if for each $\bar{x} = (x_k)_{-\infty}^{\infty}$ in $E(X), s_n(\bar{x}) \to \bar{x}$ as $n \to \infty$ with respect to $\mathcal{T}$, where $s_n(\bar{x}) = (\dots, \theta, x_{-n}, \dots, x_{-1}, x_0, x_1, \dots, x_n, \theta, \dots)$,
(v) a $GFK-$space if $(E(X), \mathcal{T})$ is complete linear metric space,
(vi) a $GC-$space if $R_n : X \to E(X),\ R_n(x) = \delta_n(x)$ is continuous for each $n \in \mathbb{Z}$.

Where $GK-$ space, $GAK-$ space, $GFK-$ space and $GC-$ space are generalized versions defined for vector valued bilateral sequences corresponding to $K-$ space, $AK-$ space, $FK-$ space and $C-$ space which are defined for scalar sequences (see Wilansky [17] and Kamthan and Gupta [**?**]).

Theorem 3.1. *Let X be a normed space and consider $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ with $Q_{\bar{\lambda},\bar{p}}$.*
(i) *$(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda},\bar{p}})$ is a total paranormed $GK-$space,*
(ii) *$(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda},\bar{p}})$ is a $GAD-$, $GAK-$ and $GC-$space,*
(iii) *if X is seperable then so is $(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda},\bar{p}})$ and*
(iv) *if X is a Banach space then $(\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}), Q_{\bar{\lambda},\bar{p}})$ is a $GFK-$space.*

Proof. Throughout the theorem $Q_{\bar{\lambda},\bar{p}}$ will be denoted by Q
(i) We prove here continuity of scalar multiplication only and other conditions for Q to be a total paranorm can be proved very easily. To prove the continuity of scalar multiplication, it is sufficient to show:
(a) $Q(\bar{x}^{(n)}) \to 0$ and $\alpha_n \to \alpha$ imply $Q(\alpha_n \bar{x}^{(n)}) \to 0$, and
(b) $\alpha_n \to 0$ imply $Q(\alpha_n \bar{x}) \to 0$ for each $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$

Let $Q(\bar{x}^{(n)}) = \left(\sum_{-\infty}^{\infty} ||\lambda_k x_k^{(n)}||^{p_k}\right)^{1/M} \to 0$ and $\alpha_n \to \alpha$ as $n \to \infty$.
Further suppose that for some $L > 0$ $|\alpha_n| \le L$, for all $n \ge 1$.
Then

$$\begin{aligned} Q(\alpha_n \bar{x}^{(n)}) &= \left(\sum_{-\infty}^{\infty} ||\lambda_k \alpha_n x_k^{(n)}||^{p_k}\right)^{1/M} \\ &= \left(\sum_{-\infty}^{\infty} |\alpha_n|^{p_k} ||\lambda_k x_k^{(n)}||^{p_k}\right)^{1/M} \\ &\le \left(\sum_{-\infty}^{\infty} L^{p_k} ||\lambda_k x_k^{(n)}||^{p_k}\right)^{1/M} \\ &= \sup(L^{p_k/M}) \left(\sum_{-\infty}^{\infty} ||\lambda_k x_k^{(n)}||^{p_k}\right)^{1/M} \\ &\le A(L) Q(\bar{x}^{(n)}) \end{aligned}$$

where $A(L) = max(1, L)$. This implies that $Q(\alpha_n \bar{x}^{(n)}) \to 0$ as $n \to \infty$. This proves (a).

Now let $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ then for $\epsilon > 0$ there exists K such that

$$\sum_{k\in\mathbb{Z}\setminus\mathbb{Z}(-K,K)} ||\lambda_k x_k^{(n)}||^{p_k} < \left(\frac{\epsilon}{2}\right)^M.$$

Further if $\alpha_n \to 0$ we can find N such that for $n \geq N$ and $|\alpha_n| \leq 1$

$$\sum_{k\in\mathbb{Z}(-K,K)} |\alpha_n|^{p_k} ||\lambda_k x_k||^{p_k} < \left(\frac{\epsilon}{2}\right)^M \text{ and } |\alpha_n| \leq 1.$$

Thus, for all $n \geq N$ we get

$$Q(\alpha_n \bar{x}) \leq \left(\sum_{k\in\mathbb{Z}(-K,K)} ||\alpha_n \lambda_k x_k||^{p_k}\right)^{1/M} + \left(\sum_{k\in\mathbb{Z}\setminus\mathbb{Z}(-K,K)} ||\lambda_k x_k||^{p_k}\right)^{1/M}$$

$$\leq \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} = \epsilon$$

and hence (b) follows.

Further each $k \in \mathbb{Z}$, the continuity of $P_k : \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p}) \to X$ where $P_k(\bar{x}) = x_k$ follows from

$$P_k(\bar{x}) = ||x_k|| \leq |\lambda_k|^{-1}(Q(\bar{x}))^{M/p_k}.$$

Thus $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a $GK-$space which proves **(i)**.

(ii) Let $\bar{x} = (x_k) \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ and $\epsilon > 0$. Then there exists K such that

$$\left(\sum_{k\in\mathbb{Z}\setminus\mathbb{Z}(-K,K)} ||\lambda_k x_k||^{p_k}\right)^{1/M} < \epsilon$$

We now easily see that

$$Q(\bar{x} - s_{K-1}(\bar{x})) = \left(\sum_{k\in\mathbb{Z}\setminus\mathbb{Z}(-K,K)} ||\lambda_k x_k||^{p_k}\right)^{1/M} < \epsilon.$$

This shows that $\Phi(X)$ is dense in $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ as well as $s_n(\bar{x}) \to \bar{x}$ as $n \to \infty$. Hence $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a $GAD-$space and also a $GAK-$space.

Now let $R_k : X \to \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$, where $R_k(x) = \delta_k(x), \quad k \in \mathbb{Z}$. Clearly continuity of $R_k, k \in \mathbb{Z}$ follows from

$$Q(R_k(x)) = Q(\delta_k(x)) \leq |\lambda_k|^{p_k/M} ||x||^{p_k/M}.$$

Hence $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a $GC-$space.

(iii) If D is a countable dense subset of X then $\Phi(D)$ will be a countable dense subset of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

(iv) If X is a Banach space then we show that $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is complete with respect to the metric induced by Q.

Let $(\bar{x}^{(n)})$ be a Cauchy sequence in $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Then for $\epsilon > 0$ there exists N such that

(3.2)
$$\left(\sum_{-\infty}^{\infty} ||\lambda_k x_k^{(n)} - \lambda_k x_k^{(m)}||^{p_k}\right)^{1/M} < \epsilon, \quad \text{for all } n, m \geq N,$$

and so for each $k \in \mathbb{Z}$

$$||x_k^{(n)} - x_k^{(m)}|| < |\lambda_k|^{-1}\epsilon^{M/p_k} < |\lambda_k|^{-1}\epsilon, \quad \text{for all } n, m \geq N.$$

This shows that for each $k \in \mathbb{Z}$, $(\bar{x_k}^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in X. Since X is complete therefore $x_k^n \to x_k \in X$ as $n \to \infty$. Let $\bar{x} = (x_k) \in X$. Since $(\bar{x}^{(n)})$ is a Cauchy sequence therefore it will be bounded with respect to Q. Suppose

$$Q(\bar{x}^{(n)}) = \left(\sum_{-\infty}^{\infty} ||\lambda_k x_k^n||^{p_k}\right)^{1/M} \leq L,$$

for some $L > 0$ and each $n \geq 1$. Now for any $t \geq 1$, we have

$$\left(\sum_{-t}^{t}||\lambda_k x_k^{(n)}||^{p_k}\right)^{1/M} \leq \left(\sum_{-\infty}^{\infty}||\lambda_k x_k^{(n)}||^{p_k}\right)^{1/M} \leq L$$

and so taking $n \to \infty$ then $t \to \infty$ we get

$$\left(\sum_{-\infty}^{\infty}||\lambda_k x_k||^{p_k}\right)^{1/M} \leq L.$$

This shows that $\bar{x} \in \ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$.

Now it remains to show that $Q(\bar{x}^{(n)} - \bar{x}) \to 0$ as $n \to \infty$. By **(3.2)**, for each fixed t we have

$$\sum_{-t}^{t}||\lambda_k x_k^{(n)} - \lambda_k x_k^{(m)}||^{p_k} < \epsilon^M.$$

Thus if in this inequality we take $m \to \infty$ first and then $t \to \infty$ we easily get

$$\left(\sum_{-\infty}^{\infty}||\lambda_k x_k^{(n)} - \lambda_k x_k||^{p_k}\right)^{1/M} \leq \epsilon, \qquad \text{for each } n \geq N$$

and so $Q(\bar{x}^{(n)} - \bar{x}) \to 0$ as $n \to \infty$ i.e., $\bar{x}^{(n)} \to \bar{x}$ with respect to Q.

This proves the completeness of $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$. Moreover $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is a $GFK-$space because $\ell(\mathbb{Z}, X, \bar{\lambda}, \bar{p})$ is complete as well as a $GK-$space.

Competing interests

The authors declare that they have no competing interests.

□

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