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An Iterative Method for Approximating the Common Solutions of Nonexpansive Semigroups, a System of Variational Inequalities, Variational Inequalities and Equilibrium Problems

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Abstract

In this work, we consider an general iterative method for finding a common element of the set of common fixed points of a one-parameter nonexpansive semigroup problem, a system of variational inequality problems, a variational inequality problem and the set of solutions of a suitable equilibrium problem in a real Hilbert space. Further we establish a strong convergence theorem based on this method. Our results improve and extend corresponding ones announced by many others.

Keywords: Nonexpansive semigroup; Equilibrium problem; Variational inequality problem; System of variational inequality problems; Projection

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1 Introduction

Throughout the paper unless otherwise stated, let H be a real Hilbert space with norm $\|.\|$ and inner product $\langle .,. \rangle$. Let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be any sequence in H, then $x_n \to x$(respectively, $x_n \rightharpoonup x$)will denote strong (respectively, weak) convergence of the sequence $\{x_n\}$.

A mapping $T : C \to C$ is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.1}$$

One parameter family $\Gamma := \{T(t) : 0 \leq t < \infty\}$ is said a (continuous) Lipschitzian semigroup on C of mappings from C into C if the following conditions are satisfied:
(1) $T(0)x = x$ for all $x \in C$;
(2) $T(s + t) = T(s)T(t)$ for all $s, t \geq 0$;
(3) for each $t > 0$, there exists a bounded measurable function $L_t : (0, \infty) \to [0, \infty)$ such that $\|T(t)x - T(t)y\| \leq L_t\|x - y\|$, $x, y \in C$;

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(4) for each $x \in C$, the mapping $T(.)x$ from $[0, \infty)$ into C is continuous.
A Lipschitzian semigroup Γ is called nonexpansive (or contractive) if $L_t = 1$ for all $t > 0$ and asymptotically nonexpansive if $\limsup_{t\to\infty} L_t \leq 1$, respectively. Let $F(\Gamma)$ denote the common fixed point set of the semigroup Γ , i.e., $F(\Gamma) := \{x \in C : T(t)x = x, \forall t > 0\}$.

Let $A : C \to H$ be a nonlinear mapping. Then A is called

(1) monotone, if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in H;$$

(2) α-strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|x - y\|^2, \quad \forall x, y \in H;$$

(3) α-inverse strongly monotone, if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in H;$$

(4) k-Lipschitz continuous, if there exists a constant $k > 0$ such that

$$\|Ax - Ay\| \leq k\|x - y\|, \quad \forall x, y \in H.$$

An operator T is strongly positive on H if there is a constant $\bar{\gamma}$ with property

$$\langle Tx, x\rangle \geq \bar{\gamma}\|x\|^2, \quad \forall x \in H.$$

Given a nonlinear mapping $\varphi : C \to H$. Then the variational inequality problem (in short, VIP) is to find $x \in C$ such that

$$\langle \varphi x, y - x\rangle \geq 0, \quad \forall y \in C. \tag{1.1}$$

The solution of VIP(1.1) is denoted by $VI(C, \varphi)$. It is well known that if φ is strongly monotone and Lipschitz continuous mapping on C then VIP(1.1) has a unique solution. There are several different approaches towards solving this problem in finite dimensional and infinite dimensional spaces see, e.g. [1-9] and the research in this direction is intensively continued.

Next we consider the following system of variational inequality problems (in short, SVIP): find $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 B_1 y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \rho_2 B_2 x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.2}$$

where $B_i : C \to C$ is a nonlinear mapping and $\rho_i > 0$ for each $i = 1, 2$. The set of solutions of SVIP(1.2) is denoted by Ψ.

Some special cases of SVIP(1.2):

(i) If $B_1 = B_2 = B$ then SVIP(1.2) is reduced to the system of problems of finding $(x^*, y^*) \in C \times C$ such that

$$\begin{cases} \langle \rho_1 B y^* + x^* - y^*, x - x^*\rangle \geq 0, & \forall x \in C, \\ \langle \rho_2 B x^* + y^* - x^*, x - y^*\rangle \geq 0, & \forall x \in C, \end{cases} \tag{1.3}$$

which is considered and studied by Verma [10].

(ii) If $x^* = y^*$ in problem (1.3) then problem (1.3) is reduced to the following classical variational inequality problem of finding $x^* \in C$ such that

$$\langle Bx^*, x - x^*\rangle \geq 0, \quad \forall x \in C.$$

In order to find the solution of SVIP(1.2), Ceng et al. [11] introduced an approximation method known as relaxed extragradient method. Some related works, we refer to see [12-14,28-33].

Let F be a bifunction from $C \times C$ to R, where R is the set of real numbers. Moudafi in [15] studies the equilibrium problem

$$to\ find\ x^* \in C \quad such\ that\ F(x^*, y) + \langle Ax^*, y - x^*\rangle \geq 0, \quad \forall y \in C$$

where A is an α-inverse strongly monotone operator. In [16-18], the authors study the mixed problem

$$to\ find\ x^* \in C \quad such\ that\ F(x^*, y) + \varphi(x^*) - \varphi(y) \geq 0, \quad \forall y \in C$$

with φ being an opportune mapping.

Here, we study the equilibrium problem

$$to\ find\ x^* \in C \quad such\ that\ F(x^*, y) + h(x^*, y) \geq 0, \quad \forall y \in C$$

that includes all previous equilibrium problems as particular cases.

In this paper motivated by the work of K.R. Kazmi and S.H. Rizvi [19] and Giuseppe Marinoa, Luigi Muglia and YongHong Yao [20], we introduce a iterative scheme for finding a common element of the set of common fixed points of a one-parameter nonexpansive semigroup, a system of variational inequality problems, a variational inequality problem and the set of solutions of suitable equilibrium problem in a real Hilbert space. We establish a strong convergence theorem based on this method. Our results extend and improve the corresponding results of K.R. Kazmi and S.H. Rizvi [19] and many others.

2 Preliminaries

Let H be a real Hilbert space. It is well known that

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle \tag{2.1}$$

and

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

Let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \quad for\ all\ y \in C. \tag{2.3}$$

P is called a metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2 \tag{2.4}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.5}$$

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2 \tag{2.6}$$

for all $x \in H$, $y \in C$. It is easy to see that

$$u \in VI(C, \varphi) \Leftrightarrow u = P_C(u - \lambda\varphi u) \quad for\ all \lambda > 0.$$

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if the graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if, for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone mapping of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e.,

$$N_C v = \{\omega \in H : \langle v - u, \omega\rangle \geq 0,\ \forall u \in C\}$$

and define

$$Tv = \begin{cases} Av + N_C v, & v \in C, \\ \emptyset, & v \notin C. \end{cases}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in VI(C, A)$(see,for example [27]).

The following lemmas will be useful for proving the convergence results of this paper.

Lemma 2.1. *([11]). For any $(x^*, y^*) \in C \times C$,$(x^*, y^*)$ is a solution of SVIP(1.2) if and only if x^* is a fixed point of the mapping $Q : C \to C$ defined by*

$$Q(x) = P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)], \quad \forall x \in C, \tag{2.7}$$

where $y^* = P_C(x^* - \mu_2 B_2 x^*)$, $\mu_i \in (0, 2\beta_i)$ and $B_i : C \to H$ is a nonlinear mapping for each $i = 1, 2$.

Lemma 2.2. *([21]). Let C be a convex closed subset of a Hilbert space H.*
Let $F : C \times C \to R$ be a bi-function such that
(f1) $F(x, x) = 0$ for all $x \in C$;
(f2) F is monotone and upper hemicontinuous in the first variable;
(f3) F is lower semicontinuous and convex in the second variable.
Let $h : C \times C \to R$ be a bi-function such that
(h1) $h(x, x) = 0$ for all $x \in C$;
(h2) h is monotone and weakly upper semicontinuous in the first variable;
(h3) h is convex in the second variable
Moreover, let us suppose that
(H) for fixed $r > 0$ and $x \in C$, there exists a bounded $D \subset C$ and $a \in D$ such that for all $z \in C \backslash D$, $-F(a, z) + h(z, a) + \frac{1}{r}\langle a - z, z - x\rangle < 0$.
For $r > 0$ and $x \in H$, let $T_r : H \to 2^C$ be a mapping defined by

$$T_r x = \{z \in C, F(z, y) + h(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\} \tag{2.8}$$

called resolvent of F and h.
Then
(1) $T_r x \neq \emptyset$;
(2) $T_r x$ is a singleton;
(3) T_r is firmly nonexpansive;
(4) $EP(F, h) = Fix(T_r)$ and it is closed and convex, where $EP(F, h) = \{z \in C, F(z, y) + h(z, y) \geq 0, \forall y \in C\}$.

Lemma 2.3. *([21]). Let us suppose that (f1)-(f3), (h1)-(h3) and (H) hold. Let $x, y \in H$, $r_1, r_2 > 0$. Then*

$$\|T_{r_2} y - T_{r_1} x\| \leq \|y - x\| + |\frac{r_2 - r_1}{r_2}|\|T_{r_2} y - y\|.$$

Remark 2.1. In the sequel, given a sequence $\{z_n\}$, we will denote with $\omega_w\{z_n\}$ the set of cluster points of $\{z_n\}$ with respect to the weak topology, i.e.

$$\omega_w\{z_n\} = \{q \in H : there\ exists\ n_k \to \infty for\ which\ z_{n_k} \rightharpoonup q\}.$$

Lemma 2.4. *(see[20,Lemma 2.5]). Suppose that the hypotheses of Lemma 2.2 are satisfied. Let $\{r_n\}$ a sequence in $(0, +\infty)$ with $\liminf_{n\to\infty} r_n > 0$. Suppose that $\{x_n\}$ is a bounded sequence. Then the following statements are equivalent and true:*
(a) if $\|x_n - T_{r_n} x_n\| \to 0$, as $n \to \infty$, the weak cluster points of $\{x_n\}$ satisfies the problem

$$F(x, y) + h(x, y) \geq 0 \quad \forall y \in C$$

i.e. $\omega_w\{x_n\} \subseteq EP(F, h)$.
(b) the demiclosedness principle holds in the sense that, if $x_n \rightharpoonup x$. and $\|x_n - T_{r_n} x_n\| \to 0$, as $n \to \infty$, then $(I - T_{r_k})x^* = 0$, for all $k \in \mathbb{N}$.

Lemma 2.5. *(Shimizu and Takahashi, [22]). Let D be a nonempty, bounded, closed and convex subset of a real Hilbert space H and let $\Im := \{T(u) : 0 \le u < \infty\}$ a nonexpansive semigroup on D, then for any $h \ge 0$,*

$$\limsup_{t\to\infty x\in D} \|T(h)(\frac{1}{t}\int_0^t T(u)xdu) - (\frac{1}{t}\int_0^t T(u)xdu)\| = 0.$$

Lemma 2.6. *([23]). Let H be a Hilbert space, C a closed convex subset of H, and $T : C \to C$ a nonexpansive mapping with $F(T) \neq \emptyset$; if $\{x_n\}$ is a sequence in C weakly converging to x and if $\{(I-T)x_n\}$ converges strongly to y, then $(I-T)x = y$.*

Lemma 2.7. *([Xu,24]). Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying*

$$a_{n+1} \le (1-\delta_n)a_n + b_n + \sigma_n, \quad n \in \mathbb{N},$$

where $\{\delta_n\}_{n=1}^{\infty} \subset (0,1)$ and $\{b_n\}_{n=1}^{\infty}$, $\{\sigma_n\}_{n=1}^{\infty}$ is a sequence in $\mathbb{R}$ such that : (i) $\lim_{n\to\infty}\delta_n = 0$ and $\sum_{n=0}^{\infty}\delta_n = \infty$, (ii) $\limsup_{n\to\infty}\frac{b_n}{\delta_n} \le 0$,$(iii)\sigma_n \ge 0, \sum_{n=0}^{\infty}\sigma_n < \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.8. *([25]). Assume that T is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\bar{\gamma} > 0$ and $0 < \rho \le \|T\|^{-1}$. Then $\|I - \rho T\| \le 1 - \rho\bar{\gamma}$.*

Lemma 2.9. *Let H be a real Hilbert space. Then the following inequalities hold:*
(1) $\|x+y\|^2 \le \|x\|^2 + 2\langle y, x+y\rangle$;
(2) $\|x+y\|^2 \ge \|x\|^2 + 2\langle y, x\rangle$,
for all $x, y \in H$.

3 Main result

Theorem 3.1. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. For each $i = 1,2$, let $A, B_i : C \to H$ be α, β_i-inverse strongly monotone mappings, respectively. Let F, h be a bi-function from $C \times C$ to R satisfying the hypotheses of Lemma 2.2. , and $\Im =: \{T(u) : 0 \le u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im) \cap EP(F,h) \cap VI(C,A) \cap \Psi \neq \emptyset$. Let $\phi : H \to H$ be a nonexpansive mapping and T be a strongly bounded linear operator on H with coefficient $\bar{\gamma}$ such that $0 < \gamma < \theta\bar{\gamma}$ and $0 < \theta \le \|T\|^{-1}$. Let $\{r_n\} \subset (0,\infty)$ be a real sequence such that $\liminf_{n\to\infty} r_n > 0$. Suppose $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \ \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A u_n), \\ y_n = \delta_n z_n + (1-\delta_n)P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \beta_n x_n + (1-\beta_n)(\frac{1}{t_n}\int_0^{t_n} T(u)[\alpha_n\gamma\phi(x_n) + (I - \alpha_n\theta T)y_n]du), \end{cases} \tag{3.1}$$

where $\mu_i \in (0, 2\beta_i)$, for each $i = 1,2$, $\lambda_n \subset (0, 2\alpha)$, and $\{\beta_n\}$, $\{\alpha_n\}$, $\{\delta_n\}$ are sequences in [0, 1]. Assume that the following conditions are satisfied:
(i) $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty}\beta_n = 0$,$\sum_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$;
(iii) $\lim_{n\to\infty}\delta_n = 0$,$\sum_{n=1}^{\infty}|\delta_{n+1} - \delta_n| < \infty$;
(iv) $\sum_{n=1}^{\infty}|r_{n+1} - r_n| < \infty$;
(v) $\lim_{n\to\infty}\frac{|t_n - t_{n-1}|}{t_n}\frac{1}{\alpha_n(1-\beta_n)} = 0$;
(vi) $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\alpha$, $\sum_{n=1}^{\infty}|\lambda_{n+1} - \lambda_n| < \infty$.
Then $\{x_n\}$ converges strongly to w, where $w := P_\Omega(\gamma\phi + (I - \theta T))w$.

Proof. First, we show that $I - \lambda_n A$ is nonexpansive. For any $x, y \in C$,

$$\begin{aligned}\|(I-\lambda_n A)x - (I-\lambda_n A)y\|^2 &= \|x - y - \lambda_n(Ax - Ay)\|^2 \\ &= \|x-y\|^2 - 2\lambda_n\langle Ax - Ay, x - y\rangle + \lambda_n^2\|Ax - Ay\|^2 \\ &\leq \|x-y\|^2 - \lambda_n(2\alpha - \lambda_n)\|Ax - Ay\|^2 \leq \|x-y\|^2. \end{aligned} \tag{3.2}$$

Similarly we can show that the mappings $(I - \mu_i B_i)$ are nonexpansive for each $i = 1, 2$.

Next, we show that $\{x_n\}$ is bounded.

Observe that $\{u_n\}$ can be re-written as $u_n = T_{r_n}(x_n)$, $n \geq 1$. Let $x^* \in \Omega$, then

$$\|u_n - x^*\| = \|T_{r_n}(x_n) - x^*\| \leq \|x_n - x^*\|. \tag{3.3}$$

Since $x^* \in \Omega$, we have

$$x^* = P_C[P_C(x^* - \mu_2 B_2 x^*) - \mu_1 B_1 P_C(x^* - \mu_2 B_2 x^*)].$$

Putting

$$y^* = P_C(x^* - \mu_2 B_2 x^*),$$

we see that

$$x^* = P_C(y^* - \mu_1 B_1 y^*). \tag{3.4}$$

Since the mapping $A : C \to H$ is α-inverse strongly monotone, we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C(u_n - \lambda_n A u_n) - P_C(x^* - \lambda_n A x^*)\|^2 \leq \|(u_n - \lambda_n A u_n) - (x^* - \lambda_n A x^*)\|^2 \\ &\leq \|(u_n - x^*) - \lambda_n(Au_n - Ax^*)\|^2 \leq \|u_n - x^*\|^2 - \lambda_n(2\alpha - \lambda_n)\|Au_n - Ax^*\|^2 \\ &\leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2. \end{aligned} \tag{3.5}$$

Setting $t'_n = P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)]$ and $v_n = P_C(z_n - \mu_2 B_2 z_n)$. It follows that

$$\begin{aligned}\|v_n - y^*\|^2 &= \|P_C(z_n - \mu_2 B_2 z_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \leq \|(z_n - \mu_2 B_2 z_n) - (x^* - \mu_2 B_2 x^*)\|^2 \\ &\leq \|z_n - x^*\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2 z_n - B_2 x^*\|^2 \leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2. \end{aligned} \tag{3.6}$$

Further, we have

$$\begin{aligned}\|t'_n - x^*\|^2 &= \|P_C(v_n - \mu_1 B_1 v_n) - P_C(y^* - \mu_1 B_1 y^*)\|^2 \\ &\leq \|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*)\|^2 \\ &\leq \|v_n - y^*\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1 v_n - B_1 y^*\|^2 \\ &\leq \|z_n - x^*\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2 z_n - B_2 x^*\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1 v_n - B_1 y^*\|^2 \quad (3.7) \\ &\leq \|z_n - x^*\|^2 \leq \|x_n - x^*\|^2. \quad (3.8) \end{aligned}$$

So

$$\begin{aligned}\|y_n - x^*\| &= \|\delta_n z_n + (1-\delta_n)P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)] - x^*\| \\ &\leq \delta_n\|z_n - x^*\| + (1-\delta_n)\|t'_n - x^*\| \\ &\leq \delta_n\|x_n - x^*\| + (1-\delta_n)\|x_n - x^*\| \leq \|x_n - x^*\|. \end{aligned} \tag{3.9}$$

Now, put $w_n = \alpha_n\gamma\phi(x_n) + (I - \alpha_n\theta T)y_n$, $n \geq 1$. So,

$$\begin{aligned}\|w_n - x^*\| &= \|\alpha_n\gamma\phi(x_n) + (I - \alpha_n\theta T)y_n - x^*\| \\ &= \|\alpha_n(\gamma\phi(x_n) - \theta T x^*) + (I - \alpha_n\theta T)(y_n - x^*)\| \\ &\leq (1 - \alpha_n\theta\bar{\gamma})\|y_n - x^*\| + \alpha_n(\|\gamma\phi(x_n) - \gamma\phi(x^*)\| + \|\gamma\phi(x^*) - \theta T x^*\|) \\ &\leq (1 - \alpha_n\theta\bar{\gamma})\|x_n - x^*\| + \alpha_n\gamma\|x_n - x^*\| + \alpha_n\|\gamma\phi(x^*) - \theta T x^*\| \\ &= [1 - \alpha_n(\theta\bar{\gamma} - \gamma)]\|x_n - x^*\| + \alpha_n(\theta\bar{\gamma} - \gamma)\frac{\|\gamma\phi(x^*) - \theta T x^*\|}{\theta\bar{\gamma} - \gamma}. \end{aligned} \tag{3.10}$$

From (3.1), we obtain

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\beta_n(x_n-x^*)+(1-\beta_n)(\frac{1}{t_n}\int_0^{t_n}[T(u)w_n-T(u)x^*]du\| \\
&\le \beta_n\|x_n-x^*\|+(1-\beta_n)\|w_n-x^*\| \\
&\le \beta_n\|x_n-x^*\|+(1-\beta_n)([1-\alpha_n(\theta\bar{\gamma}-\gamma)]\|x_n-x^*\|+\alpha_n(\theta\bar{\gamma}-\gamma)\frac{\|\gamma\phi(x^*)-\theta Tx^*\|}{\theta\bar{\gamma}-\gamma}) \\
&= (1-\alpha_n(\theta\bar{\gamma}-\gamma)(1-\beta_n))\|x_n-x^*\|+\alpha_n(\theta\bar{\gamma}-\gamma)(1-\beta_n))\frac{\|\gamma\phi(x^*)-\theta Tv\|}{\theta\bar{\gamma}-\gamma} \\
&\le \max\{\|x_n-x^*\|,\frac{1}{\theta\bar{\gamma}-\gamma}\|\gamma\phi(x^*)-\theta Tx^*\|\} \\
&\vdots \\
&\le \max\{\|x_1-x^*\|,\frac{1}{\theta\bar{\gamma}-\gamma}\|\gamma\phi(x^*)-\theta Tx^*\|\}.
\end{aligned}$$

So, $\{x_n\}$ is bounded. Hence, $\{u_n\}$,$\{z_n\}$,$\{y_n\}$,$\{\theta Ty_n\}$,$\{w_n\}$ and $\{\frac{1}{t_n}\int_0^{t_n}T(u)w_n\}$ are bounded. From (3.10), we have

$$\begin{aligned}
\|w_n-x^*\| &\le [1-\alpha_n(\theta\bar{\gamma}-\gamma)]\|x_n-x^*\|+\alpha_n(\theta\bar{\gamma}-\gamma)\frac{\|\gamma\phi(x^*)-\theta Tx^*\|}{\theta\bar{\gamma}-\gamma} \\
&\le \max\{\|x_1-x^*\|,\frac{1}{\theta\bar{\gamma}-\gamma}\|\gamma\phi(x^*)-\theta Tx^*\|\}+\frac{1}{\theta\bar{\gamma}-\gamma}\|\gamma\phi(x^*)-\theta Tx^*\| \\
&\le \|x_1-x^*\|+\frac{2}{\theta\bar{\gamma}-\gamma}\|\gamma\phi(x^*)-\theta Tx^*\|.
\end{aligned}$$

Put $D=\{\omega\in H:\|\omega-x^*\|\le\|x_1-x^*\|+\frac{2}{\theta\bar{\gamma}-\gamma}\|\gamma\phi(x^*)-\theta Tx^*\|\}$. Then D is a nonempty, bounded, closed and convex subset of H. Since $T(u)$ is nonexpansive for any $u\in[0,\infty)$, D is $T(u)$-invariant for each $u\in[0,\infty)$ and contains $\{w_n\}$. Without loss of generality, we may assume that $\Im:=T(u):0\le u<\infty$ is a nonexpansive semigroup on D. By Lemma 2.6, we get

$$\lim_{n\to\infty}\|(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)-T(h)(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)\|=0 \tag{3.11}$$

for every $h\in[0,\infty)$. Furthermore, observe that

$$\begin{aligned}
\|x_{n+1}-T(h)x_{n+1}\| &\le \|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(u)w_n du\| \\
&\quad+\|(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)-T(h)(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)\| \\
&\quad+\|T(h)(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)-T(h)x_{n+1}\| \\
&\le 2\|x_{n+1}-\frac{1}{t_n}\int_0^{t_n}T(u)w_n du\| \\
&\quad+\|(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)-T(h)(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)\| \\
&= 2\beta_n\|x_n-\frac{1}{t_n}\int_0^{t_n}T(u)w_n du\| \\
&\quad+\|(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)-T(h)(\frac{1}{t_n}\int_0^{t_n}T(u)w_n du)\|.
\end{aligned}$$

From $\lim_{n\to\infty}\beta_n = 0$ and (3.11), we get $\lim_{n\to\infty}\|x_{n+1} - T(h)x_{n+1}\| = 0$ and hence

$$\lim_{n\to\infty}\|x_n - T(h)x_n\| = 0. \tag{3.12}$$

We next show that $\lim_{n\to\infty}\|x_n - x_{n+1}\| = 0.$
From the nonexpansivity of the mapping $(I - \lambda_n A)$, we have

$$\begin{aligned}\|z_n - z_{n-1}\| &= \|P_C(u_n - \lambda_n Au_n) - P_C(u_{n-1} - \lambda_{n-1}Au_{n-1})\|\\ &\le \|(u_n - \lambda_n Au_n) - (u_{n-1} - \lambda_{n-1}Au_{n-1})\|\\ &= \|(u_n - u_{n-1}) - \lambda_n(Au_n - Au_{n-1}) + (\lambda_n - \lambda_{n-1})Au_{n-1}\|\\ &\le \|(u_n - u_{n-1}) - \lambda_n(Au_n - Au_{n-1})\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|\\ &\le \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|. \end{aligned} \tag{3.13}$$

We next estimate

$$\begin{aligned}\|t'_n - t'_{n-1}\|^2 &= \|P_C(v_n - \mu_1 B_1 v_n) - P_C(v_{n-1} - \mu_1 B_1 v_{n-1})\|^2\\ &\le \|(v_n - \mu_1 B_1 v_n) - (v_{n-1} - \mu_1 B_1 v_{n-1})\|^2\\ &\le \|v_n - v_{n-1}\|^2 - \mu_1(2\beta_1 - \mu_1)\|B_1 v_n - B_1 v_{n-1}\|^2\\ &\le \|v_n - v_{n-1}\|^2\\ &\le \|z_n - z_{n-1}\|^2 - \mu_2(2\beta_2 - \mu_2)\|B_2 z_n - B_2 z_{n-1}\|^2\\ &\le \|z_n - z_{n-1}\|^2. \end{aligned} \tag{3.14}$$

From (3.13) and (3.14), we have

$$\|t'_n - t'_{n-1}\| \le \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|. \tag{3.15}$$

We observe that

$$\begin{aligned}\|y_n - y_{n-1}\| &= \|\delta_n z_n + (1-\delta_n)t'_n - \delta_{n-1}z_{n-1} - (1-\delta_{n-1})t'_{n-1}\|\\ &= \|\delta_n(z_n - z_{n-1}) + (\delta_n - \delta_{n-1})z_{n-1} + (1-\delta_n)(t'_n - t'_{n-1})\\ &\quad + [(1-\delta_n) - (1-\delta_{n-1})]t'_{n-1}\|\\ &\le \delta_n\|z_n - z_{n-1}\| + |\delta_n - \delta_{n-1}|\|z_{n-1}\| + (1-\delta_n)\|t'_n - t'_{n-1}\| + |\delta_n - \delta_{n-1}|\|t'_{n-1}\|\\ &\le \delta_n\|z_n - z_{n-1}\| + |\delta_n - \delta_{n-1}|\|z_{n-1}\| + (1-\delta_n)\|z_n - z_{n-1}\| + |\delta_n - \delta_{n-1}|\|t'_{n-1}\|\\ &= \|z_n - z_{n-1}\| + |\delta_n - \delta_{n-1}|(\|z_{n-1}\| + \|t'_{n-1}\|)\\ &\le \|z_n - z_{n-1}\| + |\delta_n - \delta_{n-1}|M\\ &\le \|u_n - u_{n-1}\| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| + |\delta_n - \delta_{n-1}|M, \end{aligned}$$

where $M = \sup_{n\ge1}\{\|z_n\|\| + \|\|t'_n\|\}$.
On the other hand, $u_n = T_{r_n}(x_n)$ and $u_{n-1} = T_{r_{n-1}}(x_{n-1})$, by Lemma 2.3, we have

$$\|u_n - u_{n-1}\| \le \|x_n - x_{n-1}\| + L|1 - \frac{r_{n-1}}{r_n}|, \tag{3.16}$$

where $L = \sup_n\|u_n - x_n\|$. Hence

$$\|t'_n - t'_{n-1}\| \le \|x_n - x_{n-1}\| + L|1 - \frac{r_{n-1}}{r_n}| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\|, \tag{3.17}$$

and

$$\|y_n - y_{n-1}\| \le \|x_n - x_{n-1}\| + L|1 - \frac{r_{n-1}}{r_n}| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| + |\delta_n - \delta_{n-1}|M. \tag{3.18}$$

From (3.18),we have

$$\begin{aligned}\|w_n - w_{n-1}\| &= \|\alpha_n\gamma\phi(x_n) + (I - \alpha_n\theta T)y_n - \alpha_{n-1}\gamma\phi(x_{n-1}) - (I - \alpha_{n-1}\theta T)y_{n-1}\| \\ &= \|\alpha_n\gamma(\phi(x_n) - \phi(x_{n-1})) + (\alpha_n - \alpha_{n-1})\gamma\phi(x_{n-1}) + (I - \alpha_n\theta T)(y_n - y_{n-1}) \\ &+ [(I - \alpha_n\theta T) - (I - \alpha_{n-1}\theta T)]y_{n-1}\| \\ &\leq \alpha_n\gamma\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|\gamma\phi(x_{n-1})\| + (1 - \alpha_n\theta\bar{\gamma})\|y_n - y_{n-1}\| \\ &+ |\alpha_n - \alpha_{n-1}|\|\theta T y_{n-1}\| \\ &\leq \alpha_n\gamma\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|\gamma\phi(x_{n-1})\| + \|\theta T y_{n-1}\|) \\ &+ (1 - \alpha_n\theta\bar{\gamma})[\|x_n - x_{n-1}\| + L|1 - \frac{r_{n-1}}{r_n}| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| + |\delta_n - \delta_{n-1}|M] \\ &\leq [1 - \alpha_n(\theta\bar{\gamma} - \gamma)]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|\gamma\phi(x_{n-1})\| + \|\theta T y_{n-1}\|) \\ &+ L|1 - \frac{r_{n-1}}{r_n}| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| + |\delta_n - \delta_{n-1}|M. \end{aligned} \quad (3.19)$$

Let $\theta_n := \frac{1}{t_n}\int_0^{t_n} T(u)w_n du$, $n \geq 1$. Then, we have

$$\begin{aligned}\|\theta_n - \theta_{n-1}\| &= \|\frac{1}{t_n}\int_0^{t_n}[T(u)w_n - T(u)w_{n-1}]du \\ &+ (\frac{1}{t_n} - \frac{1}{t_{n-1}})\int_0^{t_{n-1}} T(u)w_{n-1}du + \frac{1}{t_n}\int_{t_{n-1}}^{t_n} T(u)w_{n-1}du\|,\end{aligned}$$

if $x^* \in \Omega$, we can write

$$\begin{aligned}\|\theta_n - \theta_{n-1}\| &= \|\frac{1}{t_n}\int_0^{t_n}[T(u)w_n - T(u)w_{n-1}]du + (\frac{1}{t_n} - \frac{1}{t_{n-1}})\int_0^{t_{n-1}}[T(u)w_{n-1} - T(u)x^*]du \\ &+ \frac{1}{t_n}\int_{t_{n-1}}^{t_n}[T(u)w_{n-1} - T(u)x^*]du\|.\end{aligned}$$

Thus,

$$\|\theta_n - \theta_{n-1}\| \leq \|w_n - w_{n-1}\| + (\frac{2|t_n - t_{n-1}|}{t_n})\|w_{n-1} - x^*\|. \quad (3.20)$$

Substituting (3.19) into (3.20), we obtain

$$\begin{aligned}\|\theta_n - \theta_{n-1}\| &\leq [1 - \alpha_n(\theta\bar{\gamma} - \gamma)]\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|\gamma\phi(x_{n-1})\| + \|\theta T y_{n-1}\|) \\ &+ L|1 - \frac{r_{n-1}}{r_n}| + |\lambda_n - \lambda_{n-1}|\|Au_{n-1}\| + |\delta_n - \delta_{n-1}|M + (\frac{2|t_n - t_{n-1}|}{t_n})\|w_{n-1} - x^*\|,\end{aligned} \quad (3.21)$$

From (3.1), we have $x_{n+1} = \beta_n x_n + (1 - \beta_n)\theta_n$ and this implies that

$$\begin{aligned}\|x_{n+1} - x_n\| &= \|\beta_n x_n + (1 - \beta_n)\theta_n - \beta_{n-1}x_{n-1} - (1 - \beta_{n-1})\theta_{n-1}\| \\ &\leq \beta_n\|x_n - x_{n-1}\| + (1 - \beta_n)\|\theta_n - \theta_{n-1}\| + |\beta_n - \beta_{n-1}|(\|x_{n-1}\| + \|\theta_{n-1}\|).\end{aligned} \quad (3.22)$$

Using (3.21) in (3.22), we obtain

$$\begin{aligned}\|x_{n+1}-x_n\| &\le \beta_n\|x_n-x_{n-1}\| + (1-\beta_n)[(1-\alpha_n(\theta\bar{\gamma}-\gamma))\|x_n-x_{n-1}\| \\ &+ |\alpha_n-\alpha_{n-1}|(\|\gamma\phi(x_{n-1})\| + \|\theta T y_{n-1}\|) \\ &+ L|1-\frac{r_{n-1}}{r_n}| + |\lambda_n-\lambda_{n-1}|\|Au_{n-1}\| + |\delta_n-\delta_{n-1}|M + (\frac{2|t_n-t_{n-1}|}{t_n})\|w_{n-1}-x^*\|] \\ &+ |\beta_n-\beta_{n-1}|(\|x_{n-1}\| + \|\theta_{n-1}\|) \\ &\le [1-(\theta\bar{\gamma}-\gamma)\alpha_n(1-\beta_n)]\|x_n-x_{n-1}\| + |\alpha_n-\alpha_{n-1}|(\|\gamma\phi(x_{n-1})\| + \|\theta T y_{n-1}\|) \\ &+ L|1-\frac{r_{n-1}}{r_n}| + |\lambda_n-\lambda_{n-1}|\|Au_{n-1}\| + |\delta_n-\delta_{n-1}|M + (\frac{2|t_n-t_{n-1}|}{t_n})\|w_{n-1}-x^*\|] \\ &+ |\beta_n-\beta_{n-1}|(\|x_{n-1}\| + \|\theta_{n-1}\|).\end{aligned}$$

If we call $D' := \max\{\sup_{n\ge1}(\|\gamma\phi(x_{n-1})\| + \|\theta T y_{n-1}\|), \sup_{n\ge1}(\|x_{n-1}\| + \|\theta_{n-1}\|), \sup_{n\ge1}\|w_{n-1} - x^*\|, L, M, \sup_{n\ge1}\|Au_{n-1}\|\}$ and $b > 0$ a minorant for $\{r_n\}$, we have

$$\begin{aligned}\|x_{n+1}-x_n\| \leqq [1-(\theta\bar{\gamma}-\gamma)\alpha_n(1-\beta_n)]\|x_n-x_{n-1}\| + D'[|\alpha_n-\alpha_{n-1}| + \frac{|r_n-r_{n-1}|}{b} \\ + |\lambda_n-\lambda_{n-1}| + |\delta_n-\delta_{n-1}| + (\frac{2|t_n-t_{n-1}|}{t_n}) + |\beta_n-\beta_{n-1}|].\end{aligned}$$

From Lemma 2.8, taking

$$\varepsilon_n = (\theta\bar{\gamma}-\gamma)\alpha_n(1-\beta_n), \quad b_n = \frac{2D'|t_n-t_{n-1}|}{t_n},$$

$$\sigma_n = D'[|\alpha_n-\alpha_{n-1}| + \frac{|r_n-r_{n-1}|}{b} + |\lambda_n-\lambda_{n-1}| + |\delta_n-\delta_{n-1}| + |\beta_n-\beta_{n-1}|],$$

by using conditions (i), (ii),(iii), (iv), (v) and (vi) it follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{3.23}$$

We show that $\lim_{n\to\infty}\|u_n-y_n\| = 0$, $\lim_{n\to\infty}\|w_n-y_n\| = 0$ and $\lim_{n\to\infty}\|x_n-u_n\| = \|x_n - T_{r_n}x_n\| = 0$. Now, by the convexity of $\|.\|^2$, we have

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\beta_n(x_n-x^*) + (1-\beta_n)(\frac{1}{t_n}\int_0^{t_n}[T(u)w_n - T(u)x^*]du\|^2 \\ &\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\|w_n-x^*\|^2 \\ &= \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\|\alpha_n\gamma\phi(x_n) + (I-\alpha_n\theta T)y_n - x^*\|^2 \\ &= \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\|\alpha_n(\gamma\phi(x_n)-\theta T x^*) + (I-\alpha_n\theta T)(y_n-x^*)\|^2 \\ &\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)[(1-\alpha_n\theta\bar{\gamma})^2\|y_n-x^*\|^2 + 2\alpha_n\langle\gamma\phi(x_n)-\theta T x^*, w_n-x^*\rangle] \\ &\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)[\|y_n-x^*\|^2 + 2\alpha_n\|\gamma\phi(x_n)-\theta T x^*\|\|w_n-x^*\|] \\ &\le \beta_n\|x_n-x^*\|^2 + (1-\beta_n)\|y_n-x^*\|^2 + 2\alpha_n M_1,\end{aligned} \tag{3.24}$$

where M_1 is an appropriate constant such that $M_1 := \sup_{n\ge1}\{\|\gamma\phi(x_n)-\theta T x^*\|\|w_n-x^*\|\}$. From

(3.7), we obtain

$$\begin{aligned}\|y_n - x^*\|^2 &\le \delta_n\|z_n - x^*\|^2 + (1-\delta_n)\|t'_n - x^*\|^2\\ &\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)\|t'_n - x^*\|^2\\ &\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)[\|z_n - x^*\|^2 - \mu_2(2\beta_2-\mu_2)\|B_2z_n - B_2x^*\|^2\\ &- \mu_1(2\beta_1-\mu_1)\|B_1v_n - B_1y^*\|^2]\\ &\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)[\|u_n - x^*\|^2 - \lambda_n(2\alpha-\lambda_n)\|Au_n - Ax^*\|^2\\ &- \mu_2(2\beta_2-\mu_2)\|B_2z_n - B_2x^*\|^2 - \mu_1(2\beta_1-\mu_1)\|B_1v_n - B_1y^*\|^2]\\ &\le \delta_n\|x_n - x^*\|^2 + (1-\delta_n)[\|x_n - x^*\|^2 - \lambda_n(2\alpha-\lambda_n)\|Au_n - Ax^*\|^2\\ &- \mu_2(2\beta_2-\mu_2)\|B_2z_n - B_2x^*\|^2 - \mu_1(2\beta_1-\mu_1)\|B_1v_n - B_1y^*\|^2]\\ &\le \|x_n - x^*\|^2 - (1-\delta_n)[\lambda_n(2\alpha-\lambda_n)\|Au_n - Ax^*\|^2\\ &+ \mu_2(2\beta_2-\mu_2)\|B_2z_n - B_2x^*\|^2 + \mu_1(2\beta_1-\mu_1)\|B_1v_n - B_1y^*\|^2]. \end{aligned} \quad (3.25)$$

Substituting (3.25) into (3.24), we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\{\|x_n - x^*\|^2 - (1-\delta_n)[(\lambda_n(2\alpha-\lambda_n)\|Au_n - Ax^*\|^2\\ &+ \mu_2(2\beta_2-\mu_2)\|B_2z_n - B_2x^*\|^2 + \mu_1(2\beta_1-\mu_1)\|B_1v_n - B_1y^*\|^2)]\} + 2\alpha_nM_1\\ &\le \|x_n - x^*\|^2 - (1-\beta_n)(1-\delta_n)[\lambda_n(2\alpha-\lambda_n)\|Au_n - Ax^*\|^2\\ &+ \mu_2(2\beta_2-\mu_2)\|B_2z_n - B_2x^*\|^2 + \mu_1(2\beta_1-\mu_1)\|B_1v_n - B_1y^*\|^2] + 2\alpha_nM_1.\end{aligned}$$

It follows that

$$\begin{aligned}&(1-\beta_n)(1-\delta_n)[\lambda_n(2\alpha-\lambda_n)\|Au_n - Ax^*\|^2 + \mu_2(2\beta_2-\mu_2)\|B_2z_n - B_2x^*\|^2\\ &+ \mu_1(2\beta_1-\mu_1)\|B_1v_n - B_1y^*\|^2]\\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \alpha_nM_1 \le \|x_n - x_{n+1}\|(\|x_n - x_{n+1}\| + \|x_{n+1} - x^*\|) + 2\alpha_nM_1.\end{aligned} \quad (3.26)$$

Since $0 < \liminf_{n\to\infty}\lambda_n \le \limsup_{n\to\infty}\lambda_n < 2\alpha$, $\|x_n - x_{n+1}\| \to 0$, $\alpha_n \to 0$, $\beta_n \to 0$ and $\delta_n \to 0$, we obtain $\lim_{n\to\infty}\|Au_n - Ax^*\| = 0$, $\lim_{n\to\infty}\|B_1v_n - B_1y^*\| = 0$, $\lim_{n\to\infty}\|B_2z_n - B_2x^*\| = 0$. By (2.1) and (2.4), we also have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C(u_n - \lambda_nAu_n) - P_C(x^* - \lambda_nAx^*)\|^2\\ &\le \langle u_n - \lambda_nAu_n - (x^* - \lambda_nAx^*), z_n - x^*\rangle\\ &\le \frac{1}{2}[\|u_n - \lambda_nAu_n - (x^* - \lambda_nAx^*)\|^2 + \|z_n - x^*\|^2 - \|u_n\\ &- x^* - \lambda_n(Au_n - Ax^*) - (z_n - x^*)\|^2]\\ &\le \frac{1}{2}[\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n - \lambda_n(Au_n - Ax^*)\|^2]\\ &\le \frac{1}{2}[\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, Au_n - Ax^*\rangle\\ &- \lambda_n^2(Au_n - Ax^*)\|^2]\\ &\le \frac{1}{2}[\|x_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|].\end{aligned}$$

Hence

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|. \quad (3.27)$$

Next, we estimate

$$\begin{aligned}
\|v_n - y^*\|^2 &= \|P_C(z_n - \mu_2 B_2 z_n) - P_C(x^* - \mu_2 B_2 x^*)\|^2 \\
&\le \langle (z_n - \mu_2 B_2 z_n) - (x^* - \mu_2 B_2 x^*), v_n - y^* \rangle \\
&\le \frac{1}{2}[\|z_n - x^* - \mu_2(B_2 z_n - B_2 x^*)\|^2 + \|v_n - y^*\|^2 \\
&- \|z_n - x^* - \mu_2(B_2 z_n - B_2 x^*) - (v_n - y^*)\|^2] \\
&\le \frac{1}{2}[\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \|z_n - v_n - \mu_2(B_2 z_n - B_2 x^*) - (x^* - y^*)\|^2] \\
&\le \frac{1}{2}[\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \|z_n - v_n - (x^* - y^*)\|^2 \\
&+ 2\mu_2\langle z_n - v_n - (x^* - y^*), B_2 z_n - B_2 x^*\rangle - \mu_2^2\|B_2 z_n - B_2 x^*\|^2] \\
&\le \frac{1}{2}[\|z_n - x^*\|^2 + \|v_n - y^*\|^2 - \|z_n - v_n - (x^* - y^*)\|^2 \\
&+ 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\|].
\end{aligned}$$

Hence

$$\begin{aligned}
\|v_n - y^*\|^2 &\le \|z_n - x^*\|^2 - \|z_n - v_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\| \\
&\le \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\| \\
&- \|z_n - v_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\|. \qquad (3.28)
\end{aligned}$$

Similarly, we also estimate

$$\begin{aligned}
\|t'_n - x^*\|^2 &= \|P_C(v_n - \mu_1 B_1 v_n) - P_C(y^* - \mu_1 B_1 y^*)\|^2 \\
&\le \langle (v_n - \mu_1 B_1 v_n) - (y^* - \mu_1 B_1 y^*), t'_n - x^* \rangle \\
&\le \frac{1}{2}[\|v_n - y^* - \mu_1(B_1 z_n - B_1 y^*)\|^2 + \|t'_n - x^*\|^2 \\
&- \|v_n - y^* - \mu_1(B_1 v_n - B_1 y^*) - (t'_n - x^*)\|^2] \\
&\le \frac{1}{2}[\|v_n - y^*\|^2 + \|t'_n - x^*\|^2 - \|v_n - t'_n - \mu_1(B_1 v_n - B_1 y^*) + (x^* - y^*)\|^2] \\
&\le \frac{1}{2}[\|v_n - y^*\|^2 + \|t'_n - x^*\|^2 - \|v_n - t'_n + (x^* - y^*)\|^2 \\
&+ 2\mu_1\langle v_n - t'_n + (x^* - y^*), B_1 v_n - B_1 y^*\rangle - \mu_1^2\|B_1 v_n - B_1 y^*\|^2] \\
&\le \frac{1}{2}[\|v_n - y^*\|^2 + \|t'_n - x^*\|^2 - \|v_n - t'_n + (x^* - y^*)\|^2 \\
&+ 2\mu_1\|v_n - t'_n + (x^* - y^*)\|\|B_1 v_n - B_1 y^*\|].
\end{aligned}$$

Hence

$$\begin{aligned}
\|t'_n - x^*\|^2 &\le \|v_n - y^*\|^2 - \|v_n - t'_n + (x^* - y^*)\|^2 + 2\mu_1\|v_n - t'_n + (x^* - y^*)\|\|B_1 v_n - B_1 y^*\| \\
&\le \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\| \\
&- \|z_n - v_n - (x^* - y^*)\|^2 + 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2 z_n - B_2 x^*\| \\
&- \|v_n - t'_n + (x^* - y^*)\|^2 + 2\mu_1\|v_n - t'_n + (x^* - y^*)\|\|B_1 v_n - B_1 y^*\|. \qquad (3.29)
\end{aligned}$$

Substituting (3.29) into (3.24), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\alpha_n M_1\\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\delta_n\|z_n - x^*\|^2 + (1-\delta_n)\|t_n' - x^*\|^2] + 2\alpha_n M_1\\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\delta_n\|z_n - x^*\|^2 + (1-\delta_n)[\|x_n - x^*\|^2 - \|u_n - z_n\|^2\\
&+ 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\| - \|z_n - v_n - (x^* - y^*)\|^2\\
&+ 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2z_n - B_2x^*\| - \|v_n - t_n' + (x^* - y^*)\|^2\\
&+ 2\mu_1\|v_n - t_n' + (x^* - y^*)\|\|B_1v_n - B_1y^*\|] + 2\alpha_n M_1\\
&\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\delta_n\|x_n - x^*\|^2 + (1-\delta_n)[\|x_n - x^*\|^2 - \|u_n - z_n\|^2\\
&+ 2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\| - \|z_n - v_n - (x^* - y^*)\|^2\\
&+ 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2z_n - B_2x^*\| - \|v_n - t_n' + (x^* - y^*)\|^2\\
&+ 2\mu_1\|v_n - t_n' + (x^* - y^*)\|\|B_1v_n - B_1y^*\|] + 2\alpha_n M_1\\
&\leq \|x_n - x^*\|^2 + (1-\beta_n)(1-\delta_n)[2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|\\
&+ 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2z_n - B_2x^*\|\\
&+ 2\mu_1\|v_n - t_n' + (x^* - y^*)\|\|B_1v_n - B_1y^*\|]\\
&- (1-\beta_n)(1-\delta_n)[\|u_n - z_n\|^2 + \|z_n - v_n - (x^* - y^*)\|^2\\
&+ \|v_n - t_n' + (x^* - y^*)\|^2] + 2\alpha_n M_1.
\end{aligned}$$

Which yields

$$\begin{aligned}
&(1-\beta_n)(1-\delta_n)[\|u_n - z_n\|^2 + \|z_n - v_n - (x^* - y^*)\|^2 + \|v_n - t_n' + (x^* - y^*)\|^2]\\
&\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + (1-\beta_n)(1-\delta_n)[2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|\\
&+ 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2z_n - B_2x^*\| + 2\mu_1\|v_n - t_n' + (x^* - y^*)\|\|B_1v_n - B_1y^*\|] + 2\alpha_n M_1\\
&= \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + (1-\beta_n)(1-\delta_n)[2\lambda_n\|u_n - z_n\|\|Au_n - Ax^*\|\\
&+ 2\mu_2\|z_n - v_n - (x^* - y^*)\|\|B_2z_n - B_2x^*\| + 2\mu_1\|v_n - t_n' + (x^* - y^*)\|\|B_1v_n - B_1y^*\|] + 2\alpha_n M_1.
\end{aligned} \tag{3.30}$$

Since $0 < \lambda_n < 2\alpha$, $\|x_n - x_{n+1}\| \to 0$, $\alpha_n \to 0$,$\beta_n \to 0$, $\delta_n \to 0$ and $\lim_{n\to\infty}\|Au_n - Ax^*\| = 0$, $\lim_{n\to\infty}\|B_1v_n - B_1y^*\| = 0$, $\lim_{n\to\infty}\|B_2z_n - B_2x^*\| = 0$. it follows from the boundedness of $\{x_n\}$, $\{u_n\}$, $\{z_n\}$, $\{t_n'\}$ and $\{v_n\}$ that

$$\lim_{n\to\infty}\|u_n - z_n\| = 0, \ \lim_{n\to\infty}\|z_n - v_n - (x^* - y^*)\| = 0, \ \lim_{n\to\infty}\|v_n - t_n' + (x^* - y^*)\| = 0. \tag{3.31}$$

Consequently, it follows that

$$\|z_n - t_n'\| \leq \|z_n - v_n - (x^* - y^*)\| + \|v_n - t_n' + (x^* - y^*)\| \to 0 (as\ n \to \infty). \tag{3.32}$$

Also $\lim_{n\to\infty}\|u_n - t_n'\| = 0$ and $\lim_{n\to\infty}\|y_n - t_n'\| = \delta_n\|z_n - t_n'\| = 0$ as $n \to \infty$

$$\|u_n - y_n\| \leq \|u_n - t_n'\| + \|t_n' - y_n\| \to 0 (as\ n \to \infty) \tag{3.33}$$

and

$$\|w_n - y_n\| = \alpha_n\|\gamma\phi(x_n) - \theta T y_n\| \to 0. \tag{3.34}$$

Next, we show that $\|x_n - u_n\| = \|x_n - T_{r_n}x_n\| \to 0$. By the firm nonexpansivity of T_{r_n} , a standard calculation (see [26]) shows that if $p \in EP(F, h)$

$$\|u_n - p\|^2 \leq \|x_n - p\|^2 - \|x_n - u_n\|^2.$$

So, let $x^* \in \Omega$; then by (3.24)

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 + 2\alpha_n M_1 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2 + 2\alpha_n M_1 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1-\beta_n)[\|x_n - x^*\|^2 - \|x_n - u_n\|^2] + 2\alpha_n M_1 \\ &= \|x_n - x^*\|^2 - (1-\beta_n)\|x_n - u_n\|^2 + 2\alpha_n M_1,\end{aligned}$$

which implies that

$$\begin{aligned}(1-\beta_n)\|x_n - u_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n M_1 \\ &\leq \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) + 2\alpha_n M_1.\end{aligned}$$

Since $\lim_{n\to\infty}\alpha_n = 0$, $\lim_{n\to\infty}\beta_n = 0$ and (3.23) , one gets

$$\lim_{n\to\infty}\|x_n - u_n\| = 0. \tag{3.35}$$

So

$$\|y_n - x_n\| \leq \|y_n - u_n\| + \|u_n - x_n\| \to 0. \tag{3.36}$$

Furthermore, from (3.33), (3.34) and (3.35), we have for every $h \in [0,\infty)$ that

$$\begin{aligned}\|T(h)w_n - T(h)x_n\| &\leq \|w_n - x_n\| \\ &\leq \|x_n - u_n\| + \|u_n - y_n\| + \|w_n - y_n\| \to 0.\end{aligned} \tag{3.37}$$

So, we obtain from (3.12) and (3.37) that

$$\|T(h)w_n - x_n\| \leq \|T(h)w_n - T(h)x_n\| + \|T(h)x_n - x_n\| \to 0.$$

Hence, we have for every $h \in [0,\infty)$ that

$$\|T(h)w_n - w_n\| \leq \|T(h)w_n - x_n\| + \|x_n - u_n\|| + \|u_n - y_n\| + \|y_n - w_n\| \to 0. \tag{3.38}$$

Next, putting $w = P_\Omega(\gamma\phi + (I-\theta T))w$, we show that $\limsup_{n\to\infty}\langle(\gamma\phi - \theta T)w, w_n - w\rangle \leq 0$. Observing that $P_\Omega(\gamma\phi + (I-\theta T))$ is a contraction. Indeed, for any $x, y \in H$, by Lemma 2.7 we have

$$\begin{aligned}\|P_\Omega(\gamma\phi + (I-\theta T))x - P_\Omega(\gamma\phi + (I-\theta T))y\| &\leq \|(\gamma\phi + (I-\theta T))x - (\gamma\phi + (I-\theta T))y\| \\ &\leq \|\gamma\phi(x) - \gamma\phi(y)\| + \|(I-\theta T))x - (I-\theta T))y\| \\ &\leq \gamma\|x-y\| + (1-\theta\bar{\gamma})\|x-y\| \\ &\leq (1-(\theta\bar{\gamma}-\gamma))\|x-y\|.\end{aligned}$$

Banachs Contraction Mapping Principle guarantees that $P_\Omega(\gamma\phi + (I-\theta T))w$ has a unique fixed point. Say $w \in H$. That is, $w = P_\Omega(\gamma\phi + (I-\theta T))w$. To do this, we choose a subsequence $\{w_{n_j}\}$ of $\{w_n\}$ such that

$$\limsup_{n\to\infty}\langle(\gamma\phi - \theta T)w, w_n - w\rangle = \lim_{j\to\infty}\langle(\gamma\phi - \theta T)w, w_{n_j} - w\rangle. \tag{3.39}$$

Since $\{w_{n_j}\}$ is bounded, without loss of generality, we can assume that $w_{n_j} \rightharpoonup q$. From (3.34), we obtain that $y_{n_j} \rightharpoonup q$. It follows from (3.33) and (3.35) that $u_{n_j} \rightharpoonup q$ and $x_{n_j} \rightharpoonup q$. Since $\{u_{n_j}\} \subset C$ and C is closed and convex, we obtain $q \in C$. Next we prove that $q \in \Omega = F(\Im) \cap EP(F,h) \cap VI(C,A) \cap \Psi$.

(a) First, we prove that $q \in \Psi$. Take any $x, y \in C$. Using (2.7), we estimate

$$\begin{aligned}\|Q(x) - Q(y)\|^2 &= \|P_C[P_C(x - \mu_2 B_2 x) - \mu_1 B_1 P_C(x - \mu_2 B_2 x)] \\ &- P_C[P_C(y - \mu_2 B_2 y) - \mu_1 B_1 P_C(y - \mu_2 B_2 y)]\|^2 \\ &\leq \|[P_C(x - \mu_2 B_2 x) - P_C(y - \mu_2 B_2 y)] \\ &- \mu_1[B_1 P_C(x - \mu_2 B_2 x) - B_1 P_C(y - \mu_2 B_2 y)]\|^2 \\ &\leq \|P_C(x - \mu_2 B_2 x) - P_C(y - \mu_2 B_2 y)\|^2 \\ &- \mu_1(2\beta_1 - \mu_1)\|B_1 P_C(x - \mu_2 B_2 x) - B_1 P_C(y - \mu_2 B_2 y)\|^2 \\ &\leq \|(x - \mu_2 B_2 x) - (y - \mu_2 B_2 y)\|^2 \leq \|x - y\|^2 \\ &- \mu_2(2\beta_2 - \mu_2)\|B_2 x - B_2 y\|^2 \leq \|x - y\|^2.\end{aligned}$$

This implies that $Q : C \to C$ is nonexpansive.

Now, we have

$$\begin{aligned}\|y_n - Q(y_n)\| &\leq \delta_n\|z_n - Q(y_n)\| + (1 - \delta_n)\|P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)] \\ &- Q(y_n)\| \\ &\leq \delta_n\|z_n - Q(y_n)\| + (1 - \delta_n)\|Q(z_n) - Q(y_n)\| \\ &\leq \delta_n\|z_n - Q(y_n)\| + (1 - \delta_n)\|z_n - y_n\|. \end{aligned} \tag{3.40}$$

Since $\delta_n \to 0$ and $\|z_n - y_n\| \to 0$ as $n \to \infty$, (3.40) implies $\liminf_{n\to\infty} \|y_n - Q(y_n)\| = 0$ and hence by Lemma 2.7, it follows that $q \in Q(q)$. Further, it follows from Lemma 2.1 that $q \in \Psi$.

(b) Next, we prove that $q \in VI(C, A)$. Indeed, let

$$Tx = \begin{cases} Ax + N_C x, & x \in C, \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x, u) \in G(T)$, $u - Ax \in N_C x$. Since $z_n \in C$, by the definition of N_C , we have

$$\langle x - z_n, u - Ax\rangle \geq 0. \tag{3.41}$$

On the other hand, since $z_n = P_C(u_n - \lambda_n A u_n)$, we have $\langle x - z_n, z_n - (u_n - \lambda_n A u_n)\rangle \geq 0$, and so

$$\langle x - z_n, \frac{z_n - u_n}{\lambda_n} + Au_n\rangle \geq 0.$$

By (3.41) and the monotonicity of A, we have

$$\begin{aligned}\langle x - z_{n_j}, u\rangle &\geq \langle x - z_{n_j}, Ax\rangle \\ &\geq \langle x - z_{n_j}, Ax\rangle - \langle x - z_{n_j}, \frac{z_{n_j} - u_{n_j}}{\lambda_{n_j}} + Au_{n_j}\rangle \\ &= \langle x - z_{n_j}, Ax - Az_{n_j}\rangle + \langle x - z_{n_j}, Az_{n_j} - Au_{n_j}\rangle - \langle x - z_{n_j}, \frac{z_{n_j} - u_{n_j}}{\lambda_{n_j}}\rangle \\ &\geq \langle x - z_{n_j}, Az_{n_j} - Au_{n_j}\rangle - \langle x - z_{n_j}, \frac{z_{n_j} - u_{n_j}}{\lambda_{n_j}}\rangle. \end{aligned} \tag{3.42}$$

Since $z_{n_j} \rightharpoonup q$ and$\|z_{n_j} - u_{n_j}\| \to 0$, we have

$$\lim_{j\to\infty} \langle x - z_{n_j}, u\rangle = \langle x - q, u\rangle \geq 0.$$

Again since T is maximal monotone, we have $q \in T^{-1}(0)$ and hence $q \in VI(C, A)$.

(c) Next, we prove that $q \in EP(F, h)$.

By $\lim_{n\to\infty} \|x_n - u_n\| = \|x_n - T_{r_n} x_n\|$ and Lemma 2.5, we note that $q \in EP(F, h)$.

(d) We show that $q \in F(\Im)$, Assume the contrary that $q \neq T(h)q$ for some $h \in [0, \infty)$. Then by Opial's condition, we obtain from (3.38) that

$$\begin{aligned}\liminf_{n\to\infty} \|z_{n_j} - q\| &< \liminf_{j\to\infty} \|z_{n_j} - T(h)q\| \\ &\le \liminf_{j\to\infty}(\|z_{n_j} - T(h)z_{n_j}\| + \|T(h)z_{n_j} - T(h)q\|) \\ &\le \liminf_{j\to\infty} \|z_{n_j} - q\|.\end{aligned}$$

This is a contradiction. Hence, $q \in F(\Im)$. Thus $q \in \Omega = F(\Im) \cap EP(F, h) \cap VI(C, A) \cap \Psi$. By (3.39) and property of metric projection, we obtain

$$\limsup_{n\to\infty}\langle(\gamma\phi - \theta T)w, w_n - w\rangle = \lim_{j\to\infty}\langle(\gamma\phi - \theta T)w, w_{n_j} - w\rangle = \langle(\gamma\phi - \theta T)w, q - w\rangle \le 0.$$

Finally, we prove that $x_n \to w$, since $w_n - w = \alpha_n(\gamma\phi(x_n) - \theta Tw) + (I - \alpha_n\theta T)(y_n - w)$

$$\begin{aligned}\|w_n - w\|^2 &= \|\alpha_n(\gamma\phi(x_n) - \theta Tw) + (I - \alpha_n\theta T)(y_n - w)\|^2 \\ &\le (1 - \alpha_n\theta\bar{\gamma})^2\|y_n - w\|^2 + 2\alpha_n\langle\gamma\phi(x_n) - \theta Tw, w_n - w\rangle \\ &\le (1 - \alpha_n\theta\bar{\gamma})^2\|x_n - w\|^2 + 2\alpha_n\langle\gamma\phi(x_n) - \gamma\phi(w), w_n - w\rangle \\ &+ 2\alpha_n\langle\gamma\phi(w) - \theta Tw, w_n - w\rangle \\ &\le (1 - \alpha_n\theta\bar{\gamma})^2\|x_n - w\|^2 + \alpha_n\gamma(\|x_n - w\|^2 + \|w_n - w\|^2) \\ &+ 2\alpha_n\langle\gamma\phi(w) - \theta Tw, w_n - w\rangle.\end{aligned}$$

This implies that

$$\begin{aligned}\|w_n - w\|^2 &\le \frac{(1 - \alpha_n\theta\bar{\gamma})^2 + \alpha_n\gamma}{1 - \alpha_n\gamma}\|x_n - w\|^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma}\langle\gamma\phi(w) - \theta Tw, w_n - w\rangle \\ &= (1 - \frac{2\alpha_n(\theta\bar{\gamma} - \gamma)}{1 - \alpha_n\gamma})\|x_n - w\|^2 + \frac{\alpha_n^2(\theta\bar{\gamma})^2}{1 - \alpha_n\gamma}\|x_n - w\|^2 \\ &+ \frac{2\alpha_n}{1 - \alpha_n\gamma}\langle\gamma\phi(w) - \theta Tw, w_n - w\rangle. \end{aligned} \tag{3.43}$$

$$\begin{aligned}\|x_{n+1} - w\|^2 &\le \beta_n\|(x_n - w)\|^2 + (1 - \beta_n)\|(\frac{1}{t_n}\int_0^{t_n}[T(u)w_n - T(u)w]du\|^2 \\ &\le \beta_n\|(x_n - w)\|^2 + (1 - \beta_n)\|w_n - w\|^2 \\ &\le \beta_n\|(x_n - w)\|^2 + (1 - \beta_n)[(1 - \frac{2\alpha_n(\theta\bar{\gamma} - \gamma)}{1 - \alpha_n\gamma})\|x_n - w\|^2 + \frac{\alpha_n^2(\theta\bar{\gamma})^2}{1 - \alpha_n\gamma}\|x_n - w\|^2 \\ &+ \frac{2\alpha_n}{1 - \alpha_n\gamma}\langle\gamma\phi(w) - \theta Tw, w_n - w\rangle] \\ &= [1 - (1 - \beta_n)\frac{2\alpha_n(\theta\bar{\gamma} - \gamma)}{1 - \alpha_n\gamma}]\|x_n - w\|^2 + \frac{\alpha_n(1 - \beta_n)}{1 - \alpha_n\gamma}[\alpha_n(\theta\bar{\gamma})^2\|x_n - w\|^2 \\ &+ 2\langle\gamma\phi(w) - \theta Tw, w_n - w\rangle] \\ &= (1 - \delta_n)\|x_n - w\|^2 + b_n.\end{aligned}$$

where $\delta_n = (1 - \beta_n)\frac{2\alpha_n(\theta\bar{\gamma} - \gamma)}{1 - \alpha_n\gamma}$ and $b_n = \frac{\alpha_n(1 - \beta_n)}{1 - \alpha_n\gamma}[\alpha_n(\theta\bar{\gamma})^2\|x_n - w\|^2 + 2\langle\gamma\phi(w) - \theta Tw, w_n - w\rangle]$. It is easily verified that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\limsup_{n\to\infty}\frac{b_n}{\delta_n} \le 0$. Then by Lemma 2.8, we obtain $\lim_{n\to\infty}\|x_n - w\| = 0$. □

Corollary 3.2. *Let C be a nonempty, closed and convex subset of a real Hilbert space H. For each $i = 1, 2$, let $A, B_i : C \to H$ be α, β_i-inverse strongly monotone mappings, respectively. Let F, h be a bi-function from $C \times C$ to R satisfying the hypotheses of Lemma 2.2 , and $\phi, Y : C \to C$ are nonexpansive mappings such that $\Omega := F(Y) \cap EP(F, h) \cap VI(C, A) \cap \Psi \neq \emptyset$. Let T be a strongly bounded linear operator on H with coefficient $\bar{\gamma}$ such that $0 < \gamma < \theta\bar{\gamma}$ and $0 < \theta \leq \|T\|^{-1}$. Let $\{r_n\} \subset (0, \infty)$ be a real sequence such that $\liminf_{n\to\infty} r_n > 0$. Suppose Let $\{x_n\}$, $\{u_n\}$, $\{y_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ z_n = P_C(u_n - \lambda_n A u_n), \\ y_n = \delta_n z_n + (1 - \delta_n) P_C[P_C(z_n - \mu_2 B_2 z_n) - \mu_1 B_1 P_C(z_n - \mu_2 B_2 z_n)], \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\frac{1}{n+1}\sum_{j=0}^{n} Y^j[\alpha_n \gamma\phi(x_n) + (I - \alpha_n\theta T)y_n]), \end{cases}$$

where $\mu_i \in (0, 2\beta_i)$, for each $i = 1, 2$, $\lambda_n \in (0, 2\alpha)$, and $\{\beta_n\}$, $\{\alpha_n\}$, $\{\delta_n\}$ are sequences in [0, 1]. Assume that the following conditions are satisfied:
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty} \beta_n = 0$,$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(iii) $\lim_{n\to\infty} \delta_n = 0$,$\sum_{n=1}^{\infty} |\delta_{n+1} - \delta_n| < \infty$;
(iv) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.
Then $\{x_n\}$ converges strongly to w, where $w := P_\Omega(\gamma\phi + (I - \theta T))w$.

Corollary 3.3. *Let C be a nonempty, closed and convex subset of a real Hilbert space H and $A : C \to H$ be α-inverse strongly monotone mappings. Let F, h be a bi-function from $C \times C$ to R satisfying the hypotheses of Lemma 2.2 , and $\Im =: \{T(u) : 0 \leq u < \infty\}$ be a one-parameter nonexpansive semigroup on H such that $\Omega := F(\Im) \cap EP(F, h) \cap VI(C, A) \neq \emptyset$. Let $\phi : H \to H$ be a nonexpansive mapping and T be a strongly bounded linear operator on H with coefficient $\bar{\gamma}$ such that $0 < \gamma < \theta\bar{\gamma}$ and $0 < \theta \leq \|T\|^{-1}$. Let $\{r_n\} \subset (0, \infty)$ be a real sequence such that $\liminf_{n\to\infty} r_n > 0$. Suppose Let $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ be sequences generated by*

$$\begin{cases} x_1 = x \in H, \\ F(u_n, y) + h(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C \\ z_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)(\frac{1}{t_n}\int_0^{t_n} T(u)[\alpha_n \gamma\phi(x_n) + (I - \alpha_n\theta T)z_n]du), \end{cases}$$

where $\lambda_n \in (0, 2\alpha)$, and $\{\beta_n\}$, $\{\alpha_n\}$ are sequences in [0, 1]. Assume that the following conditions are satisfied:
(i) $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$;
(ii) $\lim_{n\to\infty} \beta_n = 0$,$\sum_{n=1}^{\infty} |\beta_{n+1} - \beta_n| < \infty$;
(iii) $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(iv) $\lim_{n\to\infty} \frac{|t_n - t_{n-1}|}{t_n} \frac{1}{\alpha_n(1-\beta_n)} = 0$;
(v) $0 < \liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < 2\alpha$, $\sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty$.
Then $\{x_n\}$ converges strongly to w, where $w := P_\Omega(\gamma\phi + (I - \theta T))w$.

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