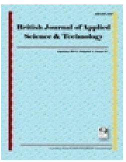

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A Survey on the Design of Lowpass Elliptic Filters

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Review Article

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Abstract

The objective of this review paper, is the presentation of the basic features of the well known class of elliptic filters. Even though this is not a new subject, the theory of the elliptic filters found in most books is restricted only to a few resulting equations, due to the great complexity associated with the Jacobian elliptic functions. The aspects of the elliptic filters described in this paper include the detailed estimation of the minimum filter order, the construction of the filter transfer function via the identification of its poles and zeros in the complex plane, as well as the application of the resulting design procedure for the construction of an elliptic filter that meets prescribed specifications.

Keywords: Elliptic filters, transfer function, poles and zeros, frequency response

2010 Mathematics Subject Classification: 00A69

1 Introduction to the Theory of Filters

Analog and digital filters are one of the most important paradigms of continuous and discrete time systems with great theoretical and practical value. At it is well known from the fundamental system theory [1][2], their time behavior is described by the impulse response $h(t)$ namely the system response to the delta (or Dirac) function $\delta(t)$, while in the domain of the complex frequency, this behavior is described by the Laplace transform of their impulse response

$$\mathcal{H}(s) = \mathcal{L}\{h(t)\} = \int_{-\infty}^{+\infty} h(t)e^{-st}dt$$

where $s = \sigma + j\omega$ is the complex frequency. The function $\mathcal{H}(s)$ is known as transfer function. If the region of convergence of the Laplace transform includes the imaginary axis, the frequency response of the filter is defined as

$$\mathcal{H}(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt$$

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and therefore, it is the continuous time Fourier transform of the impulse response $h(t)$. The frequency response of a filter can be expressed in polar form as $\mathcal{H}(j\omega) = |\mathcal{H}(j\omega)|e^{j\measuredangle\mathcal{H}(j\omega)}$ where the function $|\mathcal{H}(j\omega)|$ is known as amplitude response and the function $\measuredangle\mathcal{H}(j\omega)$ is known as phase response.

The heart of the filter design theory is the convolution theorem, according to which the convolution operation in the time domain corresponds to a multiplication in the frequency domain. It is well known that the input $x(t)$, the output $y(t)$ and the impulse response $h(t)$ of each continuous time system, are related via a convolution in the form $y(t) = h(t) * x(t)$. If the Fourier transforms of these signals are $\mathcal{Y}(j\omega)$, $\mathcal{H}(j\omega)$ and $\mathcal{X}(j\omega)$ respectively, the convolution theorem says that

$$\mathcal{Y}(j\omega) = \mathcal{H}(j\omega)\mathcal{X}(j\omega) = |\mathcal{H}(j\omega)||\mathcal{X}(j\omega)|e^{j[\measuredangle\mathcal{H}(j\omega)+\measuredangle\mathcal{X}(j\omega)]}$$

Therefore, if we want to eliminate the frequency component ω_0 of the input signal (according to Fourier analysis each signal is the superposition of an infinite number of frequency components whose frequencies are integral multiples of its fundamental frequency) we just have to set $|\mathcal{H}(j\omega_0)| = 0$. If we want to perform this elimination for an entire frequency region we just have to design the filter accordingly. According to the position of the frequency region to be eliminated in the frequency spectrum, a filter can be characterizes as low pass, high pass, band pass and band stop filter. In the case of the discrete time domain, the situation is exactly the same but there are two differences: (a) the $\mathcal{Z}$ transform is used instead of the Laplace transform with the corresponding discrete time Fourier transform to be defined if the region of convergence of the $\mathcal{Z}$ transform contains the unit circle in the complex plane and (b) the amplitude and phase response of the filter are defined up to an upper frequency of $\omega_{\text{max}} = \pi$, since this is the maximum frequency associated with a discrete signal $x[n]$ [3][4][5].

The amplitude response of an ideal filter is characterized by an absolutely constant amplitude response with a value of unity in the passband, as well as a discontinuity (namely, a step-like behavior) in the cut-off frequencies (a single frequency for low pass and high pass filters and a pair of frequencies for band pass and band stop filters). However, these filters are not causal and therefore, not realizable. Instead, in real applications a lot of different ideal filter approximations are used, that are characterized by ripples in the passband and/or in stopband as well as a transition band whose width is inverse proportional to the filter order. This order is defined as the degree of the polynomial $D(s)$ in the denominator of the rational filter transfer function whose general form is $\mathcal{H}(s) = N(s)/D(s)$ (in the circuit level, the filter order is defined as the number of the energy storage elements contained in the filter circuit). The roots of the polynomials $N(s)$ and $D(s)$ are known as the zeros and the poles of the filter transfer function and their identification allows the complete description of the filter's behavior. There are four fundamental analog filter prototypes, the Butterworth approximation characterized by the absence of ripples in the passband as well in the stopband, the Chebyshev I filter approximation whose magnitude response is characterized by ripples only in the stopband, the Chebyshev II approximation whose magnitude response is characterized by ripples only in the passband and the elliptic (or Cauer) approximation that leads to ripple behavior in the passband as well in the stopband [6].

To describe the specification of a low pass or a high pass filter design, there are four different parameters that have to be defined, namely the maximum passband loss A_p, the minimum stopband loss A_s, the passband edge ω_p and the stopband edge ω_s. On the other hand, the specifications for a band pass or a band stop filter need more parameters, even though they can be implemented via a serial combination of a low pass and a high pass filter. The parameters A_p and A_s expressed in dB units (decibels), are related to the ripple amplitudes δ_s and δ_p in the passband and the stopband via the equations [7]

$$A_p = -20\log\frac{1-\delta_p}{1+\delta_p} \qquad \text{and} \qquad A_s = -20\log\frac{\delta_s}{1+\delta_p}$$

The ratio $k = \omega_p/\omega_s$ is known as selectivity factor, while, two additional parameters that are used frequently in the filter design are defined as $\varepsilon = \sqrt{10^{A_p/10}-1}$ and $A = 10^{A_s/20}$.

The above filter characteristics are also valid for the digital filters that can be either FIR (finite impulse response) or IIR (infinite impulse response). In the fist case the filters are mainly designed via the application of an appropriate window function [8] and the frequency sampling method [3], while for the IIR case the filter can be derived by their analog prototypes using the techniques of the impulse invariance and the bilinear transformation [3]. In most cases the filter is designed to work as a low pass filter and if we need another filter type (such as a high pass, a band pass or a band stop filter), it can be derived by the analog low pass prototype via selected frequency transformations (see for example [9] for the case of digital filters). The topics covered in this paper are restricted only to the design of analog lowpass elliptic filters.

2 Elliptic Filters

Elliptic filters (also known as Cauer filters) are special types of analog and digital filters characterized by passband ripples of equal amplitude, as Chebyshev I filters as well as stopband ripples of equal amplitude as Chebyshev II filters, while, at the same time offer a very narrow transition band; however, their passband is associated with a maximal nonlinearity, regarding their phase response. The passband as well as the stopband ripple amplitude can be adjusted independently and the filter reduces to Chebyshev I or Chebyshev II filter if one of them tends to zero (note that if both amplitudes reach a zero value, the elliptic filter reduces to a Butterworth filter). Elliptic filters give the smallest filter order with respect to the other filter types for the same values of the filter design parameters. The ripples of the amplitude response of the elliptic filters in the passband and the stopband for even and odd filter order are shown in Figure 1 while the equation describing this response has the form

$$|\mathcal{H}(j\omega)| = \frac{1}{\sqrt{1+\varepsilon_p^2 \mathcal{J}_N^2(\omega,\omega_s,\omega_p,\varepsilon_s,\varepsilon_p)}}$$

where $\mathcal{J}_N(\omega,\omega_s,\omega_p,\varepsilon_s,\varepsilon_p)$ is the Jacobian elliptic function of order N defined as [7]

$$\mathcal{J}_N(\omega,\omega_s,\omega_p,\varepsilon_s,\varepsilon_p) = \mathsf{sn}\left[\kappa \mathsf{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right]$$

In the above equation, the function $\mathsf{sn}(x,k)$ is the Jacobian elliptic sine function with elliptic modulus k, the symbols ω_p and ω_s describe the passband and the stopband edge frequencies, while the parameter κ depends on the filter order N and the values of the elliptic integrals associated with the filter design. Regarding the filter order N, it depends in turn on the parameters ω's and ε's and its estimation is discussed in Section 2.1. On the other hand, the parameters ε_p and ε_s are the passband and the stopband rippple amplitudes, q is an auxiliary constant allowing the unified description of the odd ($q=0$) and even ($q=1$) order filters, while $\boldsymbol{K}_1$ is the complete elliptic integral of first kind of the elliptic module $\varepsilon_p\varepsilon_s$. It is convenient to define the elliptic modules $k_1 = \varepsilon_p\varepsilon_s$ and $k_2 = \omega_p/\omega_s$ as well as the parameters $\xi = \kappa\mathsf{sn}^{-1}(\omega/\omega_p, \omega_p/\omega_s)$ and $\zeta = \omega/\omega_p$, since, in this way, we can express the Jacobian elliptic function in the simpler form

$$R_N(\omega,\omega_s,\omega_p,\varepsilon_s,\varepsilon_p) = \mathsf{sn}[\kappa\mathsf{sn}^{-1}(\zeta,k_2) + q\boldsymbol{K}_1, k_1] = \mathsf{sn}(\xi + q\boldsymbol{K}_1, k_1)$$

As with the other filter types, the elliptic filters can be built using passive as well active circuit elements. An example circuit of each one of these approaches, is shown in Figure 2.

By adopting the usual convention and applying the basic filter theory, the parameter A_p is defined as $A_p = 10\log(1+\varepsilon_p^2)$ (dB) and therefore we have $\varepsilon_p = \sqrt{10^{A_p/10}-1}$. On the other hand, the stopband is characterized by equiripples of amplitude ε_2, with the minimun amplitude to be expressed as $1/(\varepsilon_p\varepsilon_s) = 1/k_1$. Therefore, in complete accordance with the Chebyshev II filters, we can write the equation

$$\frac{1}{A_s} = \frac{\varepsilon_s}{\sqrt{1+\varepsilon_s^2}} = \frac{1}{\sqrt{1+(1/\varepsilon_s^2)}}$$

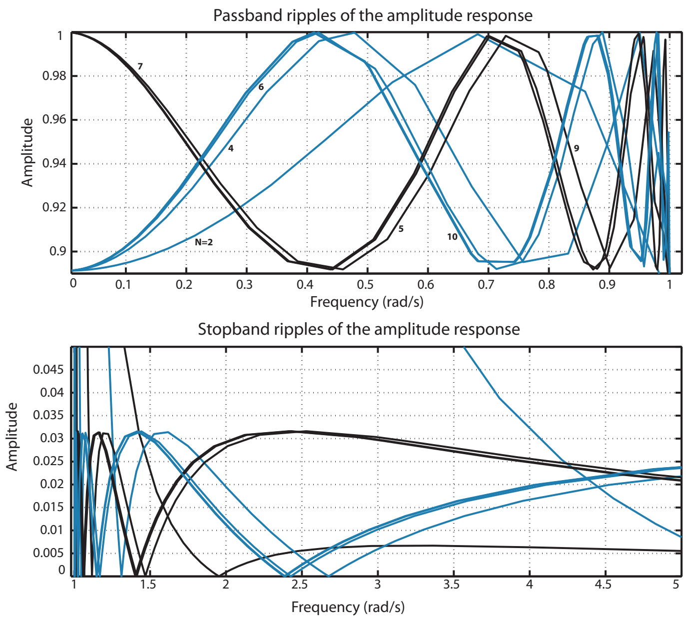

Figure 1: The passband and stopband ripples in the amplitude response of the prototype lowpass elliptic filter for parameter values $A_p = 1$ dB and $A_s = 30$ dB and for filter orders $N = 1 - 10$.

or, in dB units,

$$A_s = 10 \log \left(1 + \frac{1}{\varepsilon_s^2}\right) \qquad \text{which in turn, gives} \qquad \varepsilon_s = \frac{1}{\sqrt{10^{A_s/10} - 1}}$$

Therefore, for specific values of the A_p and A_s parameters we can use the above relations to estimate the values of ε_s and ε_p associated with the amplitude response of the elliptic filter. The graphical interpretation of these two parameters for the case of elliptic filters, is shown in Figure 6 [see also the subsection 2.5 for a description of the loss characteristic associated with these parameters].

The last comment in this introductory section is associated with the filter transfer function. At is well known from the literature, there are two main categories of ideal filter approximations, namely the polynomial and the rational approximations. In both cases, the filter gain for a filter approximation of order N has the form

$$G(\omega) = |\mathcal{H}(j\omega)| = \frac{H_0}{\sqrt{1 + \gamma^2 P_N^2(\omega)}}$$

where γ is the filter design parameter. Regarding the function $P_N(\omega)$, it is a polynomial for polynomial approximations (for example, the polynomial $P_N(\omega) = \omega^N$ for the Butterworth approximation and the Chebyshev polynomials for the Chebyshev approximations), and a rational function for rational approximations, in the form

$$P(\omega) = \frac{N(\omega)}{D(\omega)} = \frac{\alpha_N \omega^N + \alpha_{N-1}\omega^{N-1} + \alpha_{N-2}\omega^{N-2} + \cdots + \alpha_2\omega_2 + \alpha_1\omega + \alpha_0}{\beta_M \omega^M + \beta_{M-1}\omega^{M-1} + \beta_{M-2}\omega^{M-2} + \cdots + \beta_2\omega_2 + \beta_1\omega + \beta_0}$$

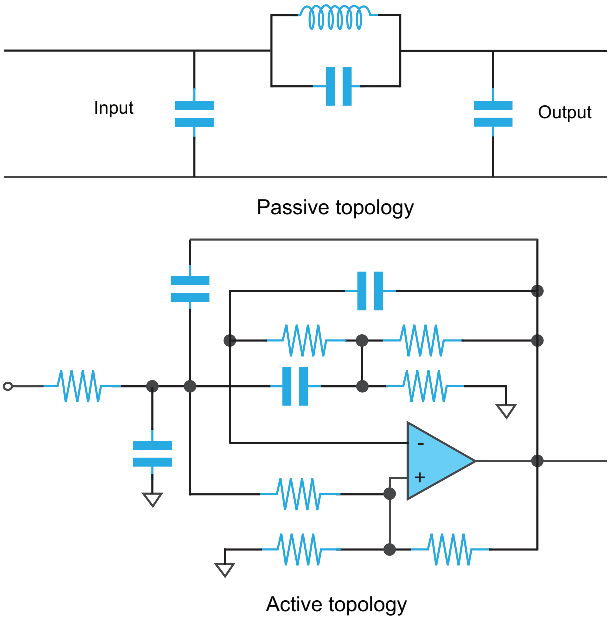

Figure 2: A typical passive and active third order elliptic analog filter.

leading to the gain function

$$G(\omega) = \frac{H_0}{\sqrt{1 + \gamma^2 \dfrac{N^2(\omega)}{D^2(\omega)}}} = \frac{H_0 D(\omega)}{\sqrt{D^2(\omega) + \gamma^2 N^2(\omega)}}$$

It should be noted that the polynomial $D(\omega)$ must be an even function of frequency, since, otherwise the well known property $D(0) = 0$ of the odd functions, would result to the vanishing of the gain function for $\omega = 0$, an unacceptable property for low pass filters. The rational approximations are characterized by the presence of zeros in the gain function, a fact that it is compatible with the fundamental theorem of Paley and Wiener. It can be proven that these zeros have the form $z_m = \pm j\omega_m$ with each pair to contribute to the numerator of the rational transfer function, the term $(s - z_m)(s - z_m^*) = (s + j\omega_m)(s - j\omega_m) = s^2 + \omega_m^2$.

Elliptic filters are rational filter approximations and therefore they are characterized by the whole set of the above properties. The detailed construction of their rational transfer function, as well as the analytic form of this function for the case of the odd and even filter orders, are presented in Section 2.4. Since the filter order N affects the number of poles and zeros, namely the number of terms appearing in the numerator and the denominator of the transfer function, it is clear that this function is strongly depended of the filter order N, a fact that also holds for all the filters approximations.

2.1 Estimating the minimum filter order

To identify the minimum filter order that meets the prescribed specifications, we start from the squared amplitude filter response at the frequency $\omega = \omega_p$

$$|\mathcal{H}(j\omega_p)|^2 = \frac{1}{1 + \varepsilon_p^2 \mathcal{J}_N^2(\omega_p, \omega_s, \omega_p, \varepsilon_s, \varepsilon_p)} = \frac{1}{1 + \varepsilon_p^2}$$

that allows the description of the Jacobian elliptic function as

$$\mathcal{J}_N^2(\omega_p, \omega_s, \omega_p, \varepsilon_s, \varepsilon_p) = \mathsf{sn}^2(\xi_{\omega=\omega_p} + q\boldsymbol{K}_1, k_1) = 1$$

or equivalently, as

$$\mathsf{sn}(\xi_{\omega=\omega_p} + q\boldsymbol{K}_1, k_1) = \pm 1$$

If we take into account the well known property $\mathsf{sn}[(2m+1)\boldsymbol{K}, k] = \pm 1$ we can write that

$$\xi_{\omega=\omega_p} + q\boldsymbol{K}_1 = (2m+1)\boldsymbol{K}_1 \qquad \kappa\mathsf{sn}^{-1}\left(\frac{\omega_p}{\omega_p}, \frac{\omega_p}{\omega_s}\right) = (2m+1-q)\boldsymbol{K}_1$$

or equivalently

$$\kappa = \frac{2m+1-q}{\mathsf{sn}^{-1}[1, (\omega_p/\omega_s)]}\boldsymbol{K}_1 = \frac{2m+1-q}{\mathsf{sn}^{-1}(1, k_2)}\boldsymbol{K}_1 = (2m+1-q)\frac{\boldsymbol{K}_1}{\boldsymbol{K}_2}$$

since, from the defining equation of the Jacobian inverse elliptic function

$$\mathsf{sn}^{-1}(\sin\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1 - k^2\sin^2\vartheta}}$$

we get

$$\mathsf{sn}^{-1}(1, k_2) = \mathsf{sn}^{-1}\left[\sin\left(\frac{\pi}{2}\right),\ k_2\right] = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1 - k_2^2\sin^2\vartheta}} = \boldsymbol{K}(k_2) = \boldsymbol{K}_2$$

On the other hand, at the frequency $\omega = \omega_s$, the amplitude response function is written as

$$|H(j\omega_s)| = \frac{1}{\sqrt{1 + \varepsilon_p^2\mathsf{sn}^2\left[\kappa\mathsf{sn}^{-1}\left(\frac{\omega_s}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right]}} = \frac{1}{A_s} = \frac{1}{\sqrt{1 + \frac{1}{\varepsilon_s^2}}}$$

and if we compare the radicand in the second and the forth expressions, we easily see that

$$\varepsilon_p^2\mathsf{sn}^2\left[\kappa\mathsf{sn}^{-1}\left(\frac{\omega_s}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = \frac{1}{\varepsilon_s^2}$$

or equivalently

$$\mathsf{sn}\left[\kappa\mathsf{sn}^{-1}\left(\frac{\omega_s}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = \frac{1}{\varepsilon_s\varepsilon_p} = \frac{1}{k_1}$$

Performing a comparison between the last equation and the expression

$$\mathsf{sn}[(2m+1)\boldsymbol{K}_1 + j\boldsymbol{K}_1', k_1] = \frac{1}{k_1}$$

we have

$$\kappa\mathsf{sn}^{-1}\left(\frac{\omega_s}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1 = \kappa\mathsf{sn}^{-1}\left(\frac{1}{k_2}, k_2\right) + q\boldsymbol{K}_1 = (2m+1)\boldsymbol{K}_1 + j\boldsymbol{K}_1'$$

and therefore

$$\mathsf{sn}^{-1}\left(\frac{1}{k_2}, k_2\right) = \frac{2m+1-q}{\kappa}\boldsymbol{K_1} + j\frac{\boldsymbol{K}_1'}{\kappa} = \boldsymbol{K}_2 + j\frac{\boldsymbol{K}_1'}{\kappa}$$

Finally, using the equation

$$\mathrm{sn}^{-1}\left(\frac{1}{k},k\right)=\boldsymbol{K}+j\boldsymbol{K}' \qquad \text{to get} \qquad \boldsymbol{K}_2+j\boldsymbol{K}_2'=\boldsymbol{K}_2+j\frac{\boldsymbol{K}_1'}{\kappa}$$

the resulting expression is

$$\kappa=(2m+1-q)\frac{\boldsymbol{K}_1}{\boldsymbol{K}_2}=\frac{\boldsymbol{K}_1'}{\boldsymbol{K}_2'}$$

The last equation is an important one in the elliptic filter design, since it provides a relationship between the parameters ω_p, ω_s, ε_p and ε_s, as a consequence of the fact that the elliptic integrals $\boldsymbol{K}_1$ and $\boldsymbol{K}_1'$ depend on the ripple amplitudes ε_p and ε_s, while the elliptic integrals $\boldsymbol{K}_2$ and $\boldsymbol{K}_2'$ depend on the frequencies ω_p and ω_s. Note the appearance of the term $2m+1-q$ in the last expression: its value is an even number if $q=1$ and an odd number if $q=0$, and in fact, it describes the filter order N. Therefore, we can set $N=2m+1-q$ to get

$$\kappa=N\frac{\boldsymbol{K}_1}{\boldsymbol{K}_2}=\frac{\boldsymbol{K}_1'}{\boldsymbol{K}_2'}$$

an expression, that allows as to express the minimum filter order associated with the values of the four filter design parameters as

$$N=\frac{\boldsymbol{K}_2}{\boldsymbol{K}_1}\frac{\boldsymbol{K}_1'}{\boldsymbol{K}_2'}$$

In most cases, the value estimated by this equation is a decimal number that has to be rounded to the next greater integer.

Even though the above relation allows the estimation of the minimum order of an elliptic filter, it is not commonly used since it involves the estimation of elliptic integrals which is not a trivial task (note, however that there is an efficient way of estimating these integrals based on the arithmetic - geometric mean) [10]. In practical applications, the value of this parameter is estimated by a more simple expression that will be constructed later in this section. The practical importance of the above relation is that it defines a restriction regarding the design specifications, in the form

$$f_1(N,\omega_p/\omega_s)=f_2(A_p,A_s)$$

that has to be satisfied by the four design filter parameters. Therefore, if we know the values of those parameters, the filter order that ensures the validity of the above relation is estimated as follows:

Starting from the expression that relates the Jacobian elliptic sine function with $\vartheta(u,q)$,

$$\mathrm{sn}[u,k]=\frac{1}{\sqrt{k}}\frac{\vartheta_1\left[\dfrac{u}{2\boldsymbol{K}},q(k)\right]}{\vartheta_4\left[\dfrac{u}{2\boldsymbol{K}},q(k)\right]}=\frac{2\sqrt[4]{q(k)}}{\sqrt{k}}\left\{\frac{\displaystyle\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(k)\sin\left[(2n+1)\frac{\pi u}{2\boldsymbol{K}}\right]}{\displaystyle 1+2\sum_{n=0}^{\infty}(-1)^n q^{n^2}(k)\cos\left(2m\frac{\pi u}{2\boldsymbol{K}}\right]}\right\}$$

and since (1) for $u=\boldsymbol{K}_1$ this function is equal to $\mathrm{sn}(\boldsymbol{K}_1,k_1)=1$ and (2) it holds that

$$\sin\left[(2n+1)\frac{\pi u}{2\boldsymbol{K}_1}\right]_{u=\boldsymbol{K}_1}=\sin\left[(2n+1)\frac{\pi}{2}\right]=\cos\left(2n\frac{\pi u}{2\boldsymbol{K}_1}\right)_{u=\boldsymbol{K}_1}=\cos(n\pi)=(-1)^n$$

we can write the expression

$$\frac{2\sqrt[4]{q(k_1)}}{\sqrt{k_1}}\left\{\sum_{n=0}^{\infty}q_1^{n(n+1)}(k_1)\right\}\bigg/\left\{1+2\sum_{n=1}^{\infty}q_1^{n^2}(k_1)\right\}=1$$

or equivalently

$$\begin{aligned} k_1 &= 4\sqrt{q\,(k_1)} \times \left[\left\{ \sum_{n=0}^{\infty} q^{n(n+1)}(k_1) \right\} \Big/ \left\{ 1 + 2\sum_{n=1}^{\infty} q^{n^2}(k_2) \right\} \right]^2 \\ &= 4\sqrt{q\,(k_1)} \times \left[\frac{1+q^2(k_1)+q^6(k_1)+q^{12}(k_1)+\cdots}{1+2q(k_1)+2q^4(k_1)+2q^9(k_1)+\cdots} \right]^2 \end{aligned}$$

where $q(k_1) = \exp(-\pi \boldsymbol{K}_1'/\boldsymbol{K}_1)$. Using the simplifications $k_1 \approx 0$ $k_1' \approx 1$ (that generally hold), we have $\boldsymbol{K}_1'/\boldsymbol{K}_1 \gg 1$ and $q(k_1) \ll 1$ and therefore the enclosed in square brackets and raised to the second power quantity, can be approximated by unity. Therefore we can set $k_1^2 \approx 4\sqrt{q(k_1)}$ and get the expression

$$k_1^2 \approx 16q(k_1) = 16\exp(-\pi \boldsymbol{K}_1'/\boldsymbol{K}_1) = 16\exp(-\pi N \boldsymbol{K}_2'/\boldsymbol{K}_2) = 16\left[\exp(-\pi \boldsymbol{K}_2'/\boldsymbol{K}_2)\right]^N = 16q^N(k_2)$$

where we used the fact that $N = \boldsymbol{K}_2\boldsymbol{K}_1'/\boldsymbol{K}_1\boldsymbol{K}_2'$. Substituting this expression in the equation

$$k_1 = \varepsilon_p\varepsilon_s = \sqrt{\frac{10^{A_p/10}-1}{10^{A_s/10}-1}} = \frac{1}{\sqrt{D}}$$

we get

$$\kappa_1^2 = \frac{10^{A_p/10}-1}{10^{A_s/10}-1} = \frac{1}{D} = 16q^N(k_2)$$

For given values of the design parameters ε_s, ε_s, A_p and A_s, the minimum filter order can be estimated by solving the above equation with respect to N. In this case, we can easily verify that

$$N = \frac{\log(16D)}{\log[1/q(k_2)]} \qquad \text{where} \qquad D = \frac{10^{A_s/10}-1}{10^{A_p/10}-1}$$

It is interesting to note, that if we know the filter order N as well as the values of the A_p and k parameters, the value of the A_s that ensures that the constraints of the design are satisfied, can be estimated from the above equation as

$$A_s = 10\log\left(\frac{10^{A_p/10}-1}{16q^N(k_2)} + 1\right)$$

2.2 Poles, zeros and min-max frequencies of the Jacobian elliptic function

By studying the form of the power response function $|\mathcal{H}(j\omega)|^2$ it can be easily noted that it gets its maximum value for frequencies such that $\mathcal{J}_N(\omega, \omega_s, \omega_p, \varepsilon_s, \varepsilon_p) = 0$ and its minimum value for frequencies such that $\mathcal{J}_N^2(\omega, \omega_s, \omega_p, \varepsilon_s, \varepsilon_p) = 1$ (J_N assumes values in $[-1, 1]$). To identify the frequency values that maximize the function $|\mathcal{H}(j\omega)|^2$ the roots of the equation

$$\text{sn}\left[\kappa\text{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = 0$$

have to be estimated, a task, that can be easily performed since the function sn(u, k) is zeroed in the values $u = 2m\boldsymbol{K}$. Therefore, we have

$$\kappa\text{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1 = 2m\boldsymbol{K}_1$$

or equivalently

$$\text{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) = \frac{(2m-q)\boldsymbol{K}_1}{\kappa} = \frac{(2m-q)\boldsymbol{K}_2}{N}$$

since $\kappa = N\boldsymbol{K}_1/\boldsymbol{K}_2$. The solution of the above equation with respect to the frequency ω is based on the equivalence $\text{sn}^{-1}(x,\kappa) = \alpha \Leftrightarrow x = \text{sn}(\alpha,\kappa)$ and leads to the result

$$\omega_m^{\text{max}} = \omega_p\,\text{sn}\left[\frac{(2m-q)\boldsymbol{K}_2}{N}, \frac{\omega_p}{\omega_s}\right]$$

for values $m = 0,1,2,\ldots,(N-1)/2$ for odd order N and $m = 0,1,2,\ldots,N/2$ for even order N. This equation allows the identification of frequencies that maximize the squared magnitude of the frequency response function in the passband.

On the other hand, the frequencies that minimize the above function, are estimated as the roots of the equation

$$\text{sn}^2\left[\kappa\text{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = 1$$

or equivalently

$$\text{sn}\left[\kappa\text{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = \pm 1$$

To proceed, we use the property $\text{sn}(u,k) = \pm 1$ where $u = (2m+1)\boldsymbol{K}$, and working in the same way we get

$$\omega_m^{\text{min}} = \omega_p\,\text{sn}\left[\frac{(2m+1-q)\boldsymbol{K}_2}{N}, \frac{\omega_p}{\omega_s}\right]$$

where $m = 0,1,2,\ldots,(N-3)/2$ for odd filter orders N and $m = 0,1,2,\ldots,(N-1)/2$ for even filter orders N. This expression allows the estimation of frequencies that minimize the squared magnitude of the frequency response function in the passband.

Let us proceed now to the estimation of the cutoff frequency ω_c in terms of the values of the parameters ω_p, ω_s, ε_p and ε_s. As it is well known, the cutoff frequency $\omega = \omega_c$ corresponds to an attenuation equal to $A_p = 3$ dB. Therefore, we have

$$|\mathcal{H}(j\omega_c)| = \frac{1}{\sqrt{2}}|\mathcal{H}(j\omega_c)|_{\text{max}} = \frac{1}{\sqrt{2}}$$

or equivalently

$$|\mathcal{H}(j\omega_c)|^2 = \frac{1}{1+\varepsilon_p^2\text{sn}^2\left[\kappa\text{sn}^{-1}\left(\frac{\omega_c}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right]} = \frac{1}{2}$$

A direct comparison allows us to write that

$$\varepsilon_p\text{sn}\left[\kappa\text{sn}^{-1}\left(\frac{\omega_c}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = 1$$

leading to the result

$$\text{sn}^{-1}\left(\frac{\omega_c}{\omega_p}, \frac{\omega_p}{\omega_s}\right) = \left[\frac{\text{sn}^{-1}(1/\varepsilon_p, \varepsilon_p\varepsilon_s) - q\boldsymbol{K}_1}{N\boldsymbol{K}_1}\right]\boldsymbol{K}_2$$

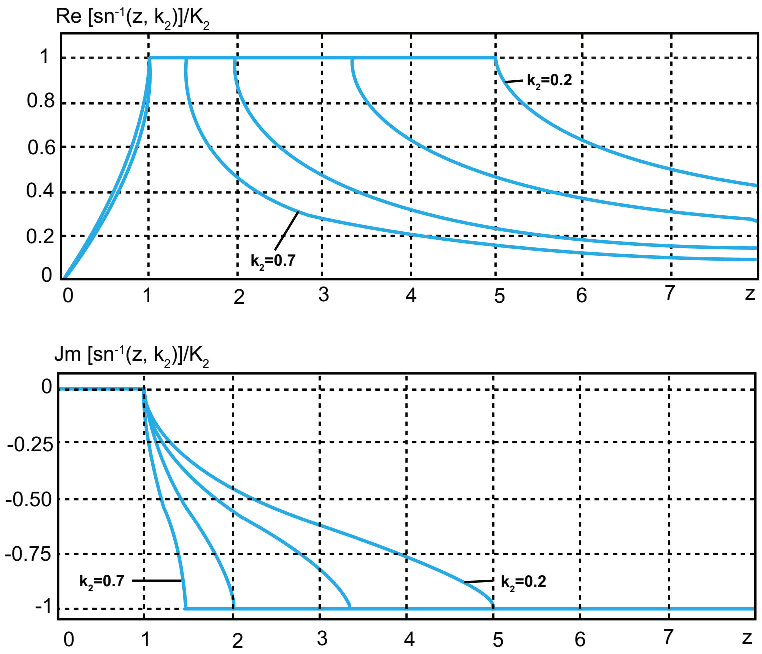

Figure 3: Variation of the real and the imaginary part of the complex function sn$^{-1}(z, k_2)$ for elliptic module values $k_2 = 0.2, 0.3, 0.5, 0.7$.

where we used again the fact that $k = N\boldsymbol{K}_1/\boldsymbol{K}_2$. The function sn$^{-1}(z, k_2)$ ($k_2 = \omega_p/\omega_s$) is a complex one, for $z = \omega_c/\omega_p > 1$, with its imaginary part

$$\Im\left\{\text{sn}^{-1}\left(\frac{\omega_c}{\omega_p}, \frac{\omega_p}{\omega_s}\right)\right\} = \Im\left\{\left[\frac{\text{sn}^{-1}(1/\varepsilon_p, \varepsilon_p\varepsilon_s)}{N\boldsymbol{K}_1}\right]\boldsymbol{K}_2\right\}$$

to vanish for $\omega_c = \omega_p$, or equivalently for $z = 1$ (this is not true for $z < 1$, or equivalently for $\omega_c < \omega_p$ as it can be seed from Figure 3 that shows the variation of the real and the imaginary part of the function sn$^{-1}(z, k_2)$ for values $k_2 = 0.2, 0.3, 0.5, 0.7$). Noting that the real part of this function is equal to $\boldsymbol{K}_2$, we can write our initial function in the form

$$\text{sn}^{-1}\left(\frac{\omega_c}{\omega_p}, \frac{\omega_p}{\omega_s}\right) = \boldsymbol{K}_2 + j\Im\left\{\left[\frac{\text{sn}^{-1}(1/\varepsilon_p, \varepsilon_p\varepsilon_s) - q\boldsymbol{K}_1}{N\boldsymbol{K}_1}\right]\boldsymbol{K}_2\right\}$$

In this case, the solution with respect the frequency ω_c will give the desired cutoff frequency as

$$\omega_c = \omega_p\,\text{sn}\left\{\boldsymbol{K}_2 + j\Im\left[\frac{\text{sn}^{-1}(1/\varepsilon_p, \varepsilon_p\varepsilon_s) - q\boldsymbol{K}_1}{N\boldsymbol{K}_1}\right]\right\}$$

Finally, let us estimate the pole frequencies of the Jacobian function $\mathcal{J}_N(\omega, \omega_s, \omega_p, \varepsilon_s, \varepsilon_p)$ that send it to infinity. Using the fact that the elliptic sine function vanishes every $2\boldsymbol{K}$, or mathematically, sn$(2m\boldsymbol{K} + j\boldsymbol{K}', k) = \pm\infty$, we can write that

$$\kappa\text{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\boldsymbol{K}_1 = 2m\boldsymbol{K}_1 + j\boldsymbol{K}_1'$$

Considering the expression

$$\text{sn}^{-1}\left(\frac{1}{k_2}, k_2\right) = \boldsymbol{K}_2 + j\boldsymbol{K}_2'$$

where $k_2 = \omega_p/\omega_s$, this equality is satisfied by the frequency $\omega_0 = \omega_s/\omega_p$. On the other hand, for frequencies $\omega > \omega_0$ this function gets a value of $x + j\boldsymbol{K}_2'$ where $x < \boldsymbol{K}_2$ and therefore, for frequencies $\omega > \omega_0$ we have $k(x + j\boldsymbol{K}_2') + q\boldsymbol{K}_1 = 2m\boldsymbol{K}_1 + j\boldsymbol{K}_1'$. Using the k value as it is expressed by its defining equation and solving with respect to the x parameter, it is found that

$$x = \frac{(2m-q)\boldsymbol{K}_2}{N} \qquad \text{and therefore} \qquad \mathsf{sn}^{-1}\left(\frac{\omega}{\omega_p}, \frac{\omega_p}{\omega_s}\right) = \frac{(2m-q)\boldsymbol{K}_2}{N} + j\boldsymbol{K}_2'$$

The poles of the Jacobian function ω are therefore given by the expression

$$\omega_m^\infty = \omega_p \mathsf{sn}\left[\frac{(2m-q)\boldsymbol{K}_2}{N} + j\boldsymbol{K}_2', \frac{\omega_p}{\omega_s}\right] = \frac{\omega_s}{\mathsf{sn}[(2m-q)\boldsymbol{K}_2/N, \omega_p/\omega_s]} = \frac{\omega_p\omega_s}{\omega_m^{\mathsf{max}}}$$

that is derived using the property $\mathsf{sn}[u + j\boldsymbol{K}', k] = 1/[k\mathsf{sn}(u, k)]$. In the above expressions, the m parameter gets the values $m = 1, 2, \ldots, (N-1)/2$ for odd filter orders and $m = 1, 2, \ldots, N/2$ for even filter orders. Note, that the Jacobian function associated with the odd order elliptic filters has an extra pole in the infinity.

The defining equations of the poles and zeros of the Jacobian elliptic function, reveal an inverse proportionality relation between them. An important consequence of this property, is that the existence of equiripples in the passband, imposes the existence of equiripples in the stopband, too.

2.3 Rationality of the Jacobian elliptic function

After the identification of the poles $\omega_p^m = \omega_m^\infty$ and the zeros $\omega_z^m = \omega_m^{\mathsf{max}}$ of the elliptic function $\mathcal{J}_N$ (this function in the literature is known as the Chebyshev rational function), it can be expressed as [10]

$$\mathcal{J}_N = D\omega^{1-q} \times \frac{\prod_{m=1}^{L}[\omega^2 - (\omega_z^m)^2]}{\prod_{m=1}^{L}[\omega^2 - (\omega_p^m)^2]} = D\omega^{1-q} \times \frac{\prod_{m=1}^{L}[\omega^2 - (\omega_z^m)^2]}{\prod_{m=1}^{L}\left[\omega^2 - \frac{\omega_p^2\omega_s^2}{(\omega_z^m)^2}\right]}$$

where the value of L is equal to $N/2$ for even orders and equal to $(N-1)/2$ for odd orders, while the scaling factor D, has the form

$$D = \begin{cases} \prod_{m=1}^{N/2}\left(\frac{\omega_p^m}{\omega_z^m}\right)^2 = \prod_{m=1}^{N/2}\left[\frac{\omega_p^2\omega_s^2}{(\omega_z^m)^4}\right] \\ \frac{\prod_{m=1}^{(N-1)/2}[(\omega_p^m)^2 - \omega_p^2]}{\omega_p\prod_{m=1}^{(N-1)/2}[\omega_p^2 - (\omega_z^m)^2]} = \omega_p \times \frac{\prod_{m=1}^{(N-1)/2}[\omega_s^2 - (\omega_z^m)^2]}{\prod_{m=1}^{(N-1)/2}(\omega_z^m)^2[\omega_p^2 - (\omega_z^m)^2]} \end{cases}$$

for even and odd filter orders respectively. This values ensure that in the passband edge $\omega = \omega_p$ as well as at the frequency $\omega = 0$ the magnitude of the function $\mathcal{J}_N$ is equal to unity. It can be proven that this function is charecterized by equirriples in the interval $[-1, +1]$ as well as in the intervals $(-\infty, -1]$ and $[+1, +\infty)$ and its plot for odd and even filter order is shown in Figure 4. The frequecy ω in the expression of the $\mathcal{J}_N$ for odd filter orders is due to the fact that the degree of the numerator polynomial for odd filter orders is equal to $N-1$ and therefore a first degree polynomial term has to be added to get a polynomial of a degree of N.

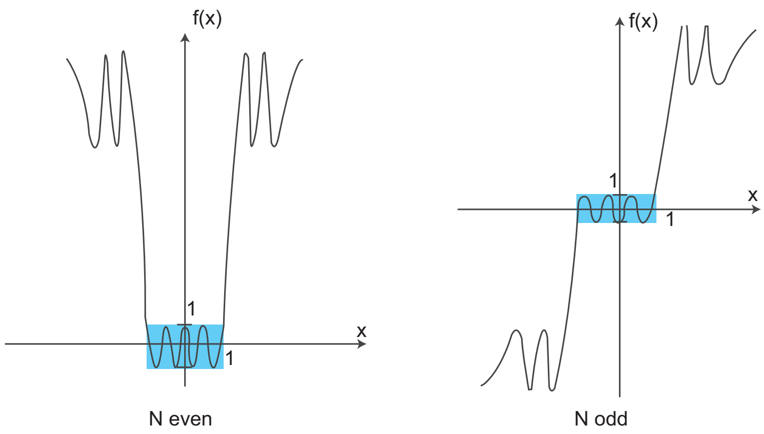

Figure 4: The Chebyshev rational function for odd and even orders.

It is interesting to note, that in practical problems the poles and zeros of the elliptic functions are not estimated from the above relations but via the next expressions, where the elliptic sinus function is expressed in terms of the functions $\vartheta_1(u,q)$, $\vartheta_4(u,q)$ and $q(k)$ [11]

$$\omega_z^m = \frac{\omega_p}{\sqrt{k}} \frac{\vartheta_1\left(\frac{2m-q}{2N}, q(k)\right)}{\vartheta_4\left(\frac{2m-q}{2N}, q(k)\right)} = \frac{\omega_p}{\sqrt{k}} \frac{\vartheta_1\left[\frac{2m-q}{2N}, \exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]}{\vartheta_4\left[\frac{2m-q}{2N}, \exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]}$$

$$= \alpha(k) \times \frac{\sum_{n=0}^{\infty}\left\{(-1)^n\left[\exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]^{n(n+1)} \sin\left[(2n+1)\pi\frac{2m-q}{2N}\right]\right\}}{1+2\sum_{n=1}^{\infty}\left\{(-1)^n\left[\exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]^{n^2} \cos\left[2n\pi\frac{2m-q}{2N}\right]\right\}}$$

$$\omega_p^m = \frac{\omega_p}{\sqrt{k}} \frac{\vartheta_1\left(\frac{2m-q}{2N}+j\frac{\boldsymbol{K_2'}}{2\boldsymbol{K}_2}, q(k)\right)}{\vartheta_4\left(\frac{2m-q}{2N}+j\frac{\boldsymbol{K_2'}}{2\boldsymbol{K}_2}, q(k)\right)} = \frac{\omega_p}{\sqrt{k}} \frac{\vartheta_1\left[\frac{2m-q}{2N}+j\frac{\boldsymbol{K_2'}}{2\boldsymbol{K}_2}, \exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]}{\vartheta_4\left[\frac{2m-q}{2N}+j\frac{\boldsymbol{K_2'}}{2\boldsymbol{K}_2}, \exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]}$$

$$= \alpha(k) \times \frac{\sum_{n=0}^{\infty}\left\{(-1)^n\left[\exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]^{n(n+1)} \sin\left[(2n+1)\pi\left(\frac{2m-q}{2N}+j\frac{\boldsymbol{K_2'}}{2\boldsymbol{K}_2}\right)\right]\right\}}{1+2\sum_{n=1}^{\infty}\left\{(-1)^n\left[\exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]^{n^2} \cos\left[2n\pi\left(\frac{2m-q}{2N}+j\frac{\boldsymbol{K_2'}}{2\boldsymbol{K}_2}\right)\right]\right\}}$$

In the above equations, the auxiliary function $\alpha(k)$ is defined as

$$\alpha(k) = \frac{2\omega_p \sqrt[4]{\exp\left(-\pi\frac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)}}{\sqrt{k}}$$

These relations do not include elliptic functions and allow the easy calculation of the poles and zeros of the elliptic function $\mathcal{J}_N$, since the arithmetic - geometric mean of two numbers $M(x,y)$ can be estimated quickly and easily, and furthermore, the functions $\vartheta_1(u,q)$, $\vartheta_4(u,q)$ and $q(k)$ are estimated via their defining equations with the accurate estimation of their value to require the calculation of a few terms only.

2.4 Poles and zeros of the transfer function

To identify the poles and zeros of the transfer function let us start by the function [10]

$$\begin{aligned}\mathcal{H}(s)\mathcal{H}(-s) &= |\mathcal{H}(j\omega)|^2_{\omega=s/j} = \frac{1}{1+\varepsilon_p^2 R_N^2(s/j,\omega_s,\omega_p,\varepsilon_s,\varepsilon_p)}\\ &= \frac{1}{1+\varepsilon_p^2 D^2(-1)^{q+1}s^{2(1-q)}\left\{\prod_{m=1}^{L}\frac{s^2+(\omega_z^m)^2}{s^2+(\omega_p^m)^2}\right\}}\\ &= \frac{\prod_{m=1}^{L}[s^2+(\omega_p^m)^2]^2}{\prod_{m=1}^{L}[s^2+(\omega_p^m)^2]^2+\varepsilon_p^2 D^2(-1)^{q+1}s^{2(1-q)}\prod_{m=1}^{L}[s^2+(\omega_p^z)^2]^2}\end{aligned}$$

whose zeros, being the roots of the algebraic equation $s^2+(\omega_p^m)^2=0$, are estimated as

$$\begin{aligned}z_m &= \pm j\omega_p^m j\omega_p \mathrm{sn}\left[\frac{(2m-q)\boldsymbol{K}_2}{N}+j\boldsymbol{K}_2', \frac{\omega_p}{\omega_s}\right]\\ &= \pm j\omega_p\frac{\omega_s}{\omega_p \mathrm{sn}\left[\frac{(2m-q)\boldsymbol{K}_2}{N}, \frac{\omega_p}{\omega_s}\right]} = \pm j\frac{\omega_s\omega_p}{\omega_z^m} = \pm\frac{j\omega_p}{\Omega_m}\end{aligned}$$

where

$$\Omega_m = \frac{1}{\omega_s}\mathrm{sn}\left[\frac{(2m-q)\boldsymbol{K}_2}{N}, \frac{\omega_p}{\omega_s}\right]$$

The above equation is valid for odd as well as even filter orders, namely for values $q=1$ and $q=0$ respectively.

On the other hand, the poles of the transfer function are the roots of the equation $1+\varepsilon_p^2 R_N^2(s/j,\omega_s,\omega_p,\varepsilon_s,\varepsilon_p)=0$ or equivalently

$$\mathrm{sn}\left[\kappa\mathrm{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right)+q\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = \frac{j}{\varepsilon_p}$$

Since the Jacobian elliptic sine function is doubly periodic with a real period of $4m\boldsymbol{K}_1$, we have

$$\mathrm{sn}\left[\kappa\mathrm{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right)+(q+4m)\boldsymbol{K}_1, \varepsilon_p\varepsilon_s\right] = \frac{j}{\varepsilon_p}$$

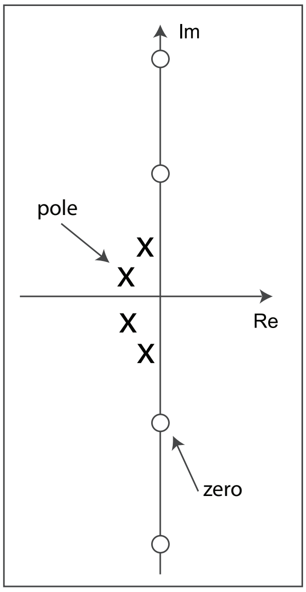

Figure 5: Pole-zero plot for the fourth order lowpass elliptic filter.

and therefore, the poles of the filter transfer function can be estimated as

$$p_m = j\omega_p \text{sn}\left[\frac{\text{sn}(j/\varepsilon_p, \varepsilon_p\varepsilon_s) - (q+4m)\boldsymbol{K}_1}{k}, \frac{\omega_p}{\omega_s}\right]$$

for values $m = 1, 2, \ldots, N$. The pole zero plot for the fourth order lowpass elliptic filter is shown in Figure 5.

2.4.1 Pole and zeros approximations

The pole defining equation can be simplified and expressed in terms of trigonometric and hyperbolic functions that are much simpler; to do this, the simplification $k_1 = \varepsilon_p\varepsilon_s \approx 0$ can be applied, that generally holds in practice. Using the results sn$(u, 0) = \sin u$ and $\boldsymbol{K}_1(0) = \pi/2$ we have

$$\kappa\,(k_1 = 0) = \frac{N}{\boldsymbol{K}_2}\boldsymbol{K}_1(0) = \frac{\pi N}{2\boldsymbol{K}_2}$$

and the pole defining equation gets the form

$$\text{sn}\left[\frac{\pi N}{2\boldsymbol{K}_2}\text{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\frac{\pi}{2}, 0\right] = \sin\left[\frac{\pi N}{2\boldsymbol{K}_2}\text{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right) + q\frac{\pi}{2}\right] = \frac{j}{\varepsilon_p}$$

At this point we distinguish between two cases:

N is odd In this case we have $q = 0$ and the last equation gets the form

$$\sin\left[\frac{\pi N}{2\boldsymbol{K}_2}\mathsf{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right)\right] = \frac{j}{\varepsilon_p}$$

or equivalently

$$-j\sin\left[\frac{\pi N}{2\boldsymbol{K}_2}\mathsf{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right)\right] = \frac{1}{\varepsilon_p}$$

From the mathematical analysis we know that $j\sin u = \sinh(ju)$ and therefore we have

$$-\sinh\left[j\frac{\pi N}{2\boldsymbol{K}_2}\mathsf{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right)\right] = \frac{1}{\varepsilon_p}$$

or

$$j\frac{\pi N}{2\boldsymbol{K}_2}\mathsf{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right) = \sinh^{-1}\left(-\frac{1}{\varepsilon_p}\right) = -\sinh^{-1}\left(\frac{1}{\varepsilon_p}\right)$$

Using the identity $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$, the right part of the equation is expressed as

$$\sinh^{-1}\left(\frac{1}{\varepsilon_p}\right) = \ln\left(\frac{1}{\varepsilon_p} + \sqrt{\frac{1}{\varepsilon_p^2}+1}\right) = \ln\left(\frac{1}{\varepsilon_p} + \sqrt{\frac{1+\varepsilon_p^2}{\varepsilon_p^2}}\right) = \ln\frac{1+\sqrt{1+\varepsilon_p^2}}{\varepsilon_p}$$

and since $\varepsilon_p = \sqrt{10^{A_p/10}-1}$ we get the expression

$$\begin{aligned}\sinh^{-1}\left(\frac{1}{\varepsilon_p}\right) &= \ln\frac{1+\sqrt{1+10^{A_p/10}-1}}{\sqrt{10^{A_p/10}-1}} = \ln\frac{1+10^{A_p/20}}{\sqrt{10^{A_p/20}+1}\sqrt{10^{A_p/20}-1}} \\ &= \ln\sqrt{\frac{10^{A_p/20}+1}{10^{A_p/20}-1}} = \frac{1}{2}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}\end{aligned}$$

Therefore we have

$$j\frac{\pi N}{2\boldsymbol{K}_2}\mathsf{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right) = -\frac{1}{2}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}$$

or equivalently,

$$\mathsf{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right) = j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}$$

and solving with respect to s,

$$\begin{aligned} s &= j\omega_p \text{sn}\left[j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}, \frac{\omega_p}{\omega_s}\right] \\ &= j\omega_p\sqrt{\frac{\omega_s}{\omega_p}}\frac{2\sqrt[4]{q(k_2)}\displaystyle\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(k_2)\sin\left[j(2n+1)\frac{\boldsymbol{K}_2}{N\pi}\pi\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}\right]}{1+2\displaystyle\sum_{n=1}^{\infty}(-1)^n q^{n^2}(k_2)\cos\left[2\pi nj\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}\right]} = \\ &= -\sqrt{\omega_s\omega_p}\frac{2\sqrt[4]{q(k_2)}\displaystyle\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(k_2)\sinh\left[(2n+1)\frac{\boldsymbol{K}_2}{N}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}\right]}{1+2\displaystyle\sum_{n=1}^{\infty}(-1)^n q^{n^2}(k_2)\cosh\left[2n\frac{\boldsymbol{K}_2}{N}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}\right]} = p_0 \end{aligned}$$

where we used the identities $\sin(ju) = j\sinh u$ and $\cos(ju) = \cosh u$. It is clear that the $s \equiv p_0$ value derived above, is not a complex but a real number. Therefore, the odd order elliptic low pass filters have a real pole $s = p_0$ that does not appear to the even order filters, even though this pole appears in the pole equations of the last filter type.

On the other hand, to identify the remaining poles of the transfer function, the periodicity of the Jacobian elliptic sine has to be used; this property is described as

$$\text{sn}(u + 4\boldsymbol{K}, k) = \text{sn}(u, k)$$

and results to a fundamental period of sn$(\alpha u, k)$ equal to $T = 4\boldsymbol{K}/\alpha$. In this case, the multiplicative coefficient of $u = \text{sn}^{-1}(s/j\omega_p, \omega_p/\omega_s)$ is $\alpha = N\boldsymbol{K}_1/\boldsymbol{K}_2 = N\pi/2\boldsymbol{K}_2$, a fact that can be easily verified noting that $\boldsymbol{K}_1(0) = \pi/2$. Therefore, the period of the elliptic function that appears in the pole defining equation is $T = 4\boldsymbol{K}_1/\alpha = 2\pi/(N\pi/2\boldsymbol{K}_2) = 4\boldsymbol{K}_2/N$, allowing us to say that if the expression

$$z_0 = j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}$$

is a solution of this equation, then the same is true for the quantities

$$z_m = z_0 + \frac{4m\boldsymbol{K}_2}{N}, \qquad \text{where} \qquad z \equiv \text{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right)$$

In other words, we can write that

$$\text{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right) = j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1} + 4\frac{\boldsymbol{K}_2}{N}m$$

and therefore, the poles of the filter transfer function can be estimated as

$$p_m = \sigma_m + j\omega_m = j\omega_p \text{sn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1} + 4\frac{\boldsymbol{K}_2}{N}m, \frac{\omega_p}{\omega_s}\right)$$

for the values $m = 0, 1, 2, \ldots, N-1$ as it can easily be verified by solving the above equation with respect to s. The value $m = 0$ is associated with the real pole p_0, while the other values of m are used to estimate the remaining $N-1$ poles. Noting that $\mathrm{sn}(u+2\boldsymbol{K},k) = -\mathrm{sn}(u,k)$ and $\mathrm{sn}(u+4\boldsymbol{K},k) = \mathrm{sn}(u,k)$, we have

$$p_m = \sigma_m + j\omega_m = j\omega_p(-1)^m \mathrm{sn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1} \pm 2\frac{\boldsymbol{K}_2}{N}m, \frac{\omega_p}{\omega_s}\right)$$

for the values $m = 0, 1, 2\ldots, (N-1)/2$.

To identify the elliptic function in the above equation, we can use the sum property of the Jacobian elliptic sinus

$$\mathrm{sn}(u+\upsilon,k) = \frac{\mathrm{sn}(u,k)\mathrm{cn}(\upsilon,k)\mathrm{dn}(\upsilon,k) \pm \mathrm{sn}(\upsilon,k)\mathrm{cn}(u,k)\mathrm{dn}(u,k)}{1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(\upsilon,k)}$$

for parameter values

$$u = j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}, \qquad \upsilon = \frac{2\boldsymbol{K}_2}{N}m, \qquad k = k_2 = \frac{\omega_p}{\omega_s}$$

Starting from the estimation of the denominator which is much more simpler and using the equation

$$p_0 = j\omega_p \mathrm{sn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}, \frac{\omega_p}{\omega_s}\right)$$

we get

$$1-k^2\mathrm{sn}^2(u,k)\mathrm{sn}^2(\upsilon,k) = 1-k^2\mathrm{sn}^2\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}, \frac{\omega_p}{\omega_s}\right)\mathrm{sn}^2\left(\frac{2\boldsymbol{K}_2}{N}m, \frac{\omega_p}{\omega_s}\right) =$$
$$1+\frac{\omega_p^2}{\omega_s^2}\frac{p_0^2}{\omega_p^2}\mathrm{sn}^2\left(\frac{2\boldsymbol{K}_2}{N}m, \frac{\omega_p}{\omega_s}\right) = 1+p_0^2\left[\frac{1}{\omega_s}\mathrm{sn}\left(\frac{2\boldsymbol{K}_2}{N}m, \frac{\omega_p}{\omega_s}\right)\right] = 1+p_0^2\Omega_m^2$$

where

$$\Omega_m = \frac{1}{\omega_s}\mathrm{sn}\left[\frac{2\boldsymbol{K}_2}{N}m, \frac{\omega_p}{\omega_s}\right] = \frac{1}{\sqrt{\omega_s\omega_p}}\frac{2\sqrt[4]{q(k_2)}\displaystyle\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}(k_2)\sin\left[(2n+1)\frac{\pi m}{N}\right]}{1+2\displaystyle\sum_{n=1}^{\infty}(-1)^n q^{n^2}(k_2)\cos\left(\frac{2n\pi m}{N}\right)}$$

for the values $m = 1, 2, \ldots, (N-1)/2$.

To estimate the expressions in the numerator we combine the properties

$$\mathrm{sn}^2(u,k)+\mathrm{cn}^2(u,k) = 1 \qquad \text{and} \qquad k^2\mathrm{sn}^2(u,k)+\mathrm{dn}^2(u,k) = 1$$

to construct the auxiliary function

$$\mathrm{cn}(u,k)\mathrm{dn}(u,k) = \sqrt{[1-\mathrm{sn}^2(u,k)][1-k^2\mathrm{sn}^2(u,k)]}$$

Therefore we have

$$\text{sn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)\text{cn}\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)\text{dn}\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)=$$
$$\text{sn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)\quad\times$$
$$\times\sqrt{\left[1-\text{sn}^2\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)\right]\left[1-k^2\text{sn}^2\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)\right]}$$

and using the defining equation of the parameters p_0 and Ω_m we get

$$\text{sn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)\text{cn}\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)\text{dn}\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)$$
$$=\frac{p_0}{j\omega_p}\sqrt{(1-\omega_p^2\Omega_m^2)(1-\omega_s^2\Omega_m^2)}=\frac{p_0}{j\omega_p}V_m$$

where

$$V_m=\sqrt{(1-\omega_p^2\Omega_m^2)(1-\omega_s^2\Omega_m^2)},\quad m=1,2,\ldots,\frac{N-1}{2}$$

For the second expression in the numerator we have

$$\text{sn}\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)\text{cn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)\text{dn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)$$
$$=\text{sn}\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)\sqrt{1-\text{sn}^2\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)}$$
$$\sqrt{1-k^2\text{sn}^2\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)}$$

Using the expressions of p_0 and Ω_m we get

$$\text{sn}\left(\frac{2\boldsymbol{K}_2}{N}m,\frac{\omega_p}{\omega_s}\right)\text{cn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)\text{dn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1},\frac{\omega_p}{\omega_s}\right)$$
$$=\omega_s\Omega_m\sqrt{\left(1+\frac{p_0^2}{\omega_p^2}\right)\left(1+\frac{p_0^2}{\omega_s^2}\right)}=\omega_s\Omega_m W$$

where

$$W=\sqrt{\left(1+\frac{p_0^2}{\omega_p^2}\right)\left(1+\frac{p_0^2}{\omega_s^2}\right)}$$

Substituting the above expressions in the defining equation of the poles p_m, it gets the form

$$p_m=\sigma_m+j\omega_m\;=\;j\omega_p(-1)^m\frac{(p_0/j\omega_p)V_m\pm\omega_s\Omega_m V}{1+p_0^2\Omega_m^2}=$$
$$=\;\frac{(-1)^m p_0V_m\pm j\omega_p\omega_s(-1)^m\Omega_m W}{1+p_0^2\Omega_m^2}=\frac{(-1)^m p_0V_m\pm j\omega_p\omega_s\Omega_m W}{1+p_0^2\Omega_m^2}$$

for the values $m = 1, 2, \ldots, N/2$. This is the final result. Therefore, the transfer function has a negative real pole p_0 contributing the term $s + p_0$, as well as $(N-1)/2$ pairs of complex conjugate poles $\sigma_m \pm j\omega_m$ each one of them contributes the term

$$\begin{aligned}\frac{1}{(s-p_m)(s-p_m^*)} &= \frac{1}{s^2-(p_m+p_m^*)s+p_m p_m^*} = \frac{1}{s^2-2\sigma_m s+\sigma_m^2+\omega_m^2} \\ &= \frac{1}{s^2+\dfrac{2p_0V_m}{1+p_0^2\Omega_m^2}s+\dfrac{(p_0V_m)^2+(\omega_p\omega_s\Omega_m W)^2}{(1+p_0^2\Omega_m^2)^2}}\end{aligned}$$

The transfer function is also characterized by complex conjugate pairs of zeros in the locations $z_m = \pm j\omega_p/\Omega_m$ ($m = 1, 2, \ldots, (N-1)/2$) with each pair to contribute the term

$$(s-z_m)(s-z_m^*) = \left(1-\frac{j\omega_p}{\omega_m}\right)\left(1+\frac{j\omega_p}{\omega_m}\right) = s^2+\frac{\omega_p^2}{\Omega_m^2}$$

Based on the above results, the transfer function of the low pass elliptic filter of odd order is expressed as

$$\mathcal{H}(s) = \frac{H_0}{s+p_0}\prod_{m=1}^{(N-1)/2}\left\{\left[s^2+\frac{\omega_p^2}{\Omega_m^2}\right]\Bigg/\left[s^2+\frac{2p_0V_m}{1+p_0^2\Omega_m^2}s+\frac{(p_0V_m)^2+(\omega_p\omega_s\Omega_m W)^2}{(1+p_0^2\Omega_m^2)^2}\right]\right\}$$

The constant H_0 can be estimated from the normalization requirement

$$R_N(\omega, \omega_p, \omega_s, \varepsilon_p, \varepsilon_s) = 1$$

for odd filter order N, leading to the value $\mathcal{H}(0) = 1$. Therefore,

$$\mathcal{H}(0) = \frac{H_0}{p_0}\prod_{m=1}^{(N-1)/2}\left\{\frac{\dfrac{\omega_p^2}{\Omega_m^2}}{\dfrac{(p_0V_m)^2+(\omega_p\omega_s\Omega_m W)^2}{(1+p_0^2\Omega_m^2)^2}}\right\} = 1$$

and the constant H_0 is defined as

$$H_0 = p_0\prod_{m=1}^{(N-1)/2}\left\{\frac{\dfrac{(p_0V_m)^2+(\omega_p\omega_s\Omega_m W)^2}{(1+p_0^2\Omega_m^2)^2}}{\dfrac{\omega_p^2}{\Omega_m^2}}\right\}$$

N **even** In this case we have $q = 1$, and therefore,

$$\sin\left[\frac{\pi N}{2\boldsymbol{K}_2}\mathrm{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right)+\frac{\pi}{2}\right] = \frac{j}{\varepsilon_p}$$

Working in the same way we get the expression

$$\mathrm{sn}^{-1}\left(\frac{s}{j\omega_p}, \frac{\omega_p}{\omega_s}\right) = j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}-\frac{\boldsymbol{K}_2}{N}$$

Since the elliptic function involved in the pole equation has a fundamental period of $4\boldsymbol{K}_2/N$ we have

$$z_m = z_0 \pm \frac{2m-1}{N}\boldsymbol{K_2}$$

and therefore, the pole equation of the filter transfer function is

$$p_m = \sigma_m + j\omega_m = j\omega_p(-1)^m \text{sn}\left(j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1} \pm \frac{2m-1}{N}\boldsymbol{K}_2, \frac{\omega_p}{\omega_s}\right)$$

In this case, we have $m = 1, 2, \ldots, N/2$ and the filter function does not have a single real pole but only a number of pairs of complex conjugate poles.

The subsequent analysis is the same as previously and the final result is

$$p_m = \pm(\sigma_m + j\omega_m)$$

where

$$\sigma_m + j\omega_m = \frac{\pm[p_0 V_m + j\omega_p\omega_s(-1)^m\Omega_m W]}{1 + p_0^2\Omega_m^2}$$

The parameters V_m and W have already been defined, while Ω_m is defined as

$$\Omega_m = \frac{1}{\omega_s}\text{sn}\left[\frac{2m-1}{N}\boldsymbol{K}_2, \frac{\omega_p}{\omega_s}\right]$$

Therefore, the transfer function for even order elliptic filters has the form

$$\mathcal{H}(s) = H_0 \prod_{m=1}^{N/2}\left\{\left[s^2 + \frac{\omega_p^2}{\Omega_m^2}\right] \Big/ \left[s^2 + \frac{2p_0V_m}{1+p_0^2\Omega_m^2}s + \frac{(p_0V_m)^2 + (\omega_p\omega_s\Omega_m W)^2}{(1+p_0^2\Omega_m^2)^2}\right]\right\}$$

with the estimation of the H_0 constant to rely on the requirement $G(0) = 10^{-A_p/20}$, as in the case of the Chebyshev I filters. Therefore we have

$$\mathcal{H}(0) = H_0 \prod_{m=1}^{(N-1)/2}\left\{\frac{\frac{\omega_p^2}{\Omega_m^2}}{\frac{(p_0V_m)^2 + (\omega_p\omega_s\Omega_m W)^2}{(1+p_0^2\Omega_m^2)^2}}\right\} = 10^{-0.05A_p}$$

and the value of H_0 is estimated as

$$H_0 = 10^{-0.05A_p} \prod_{m=1}^{(N-1)/2}\left\{\frac{\frac{(p_0V_m)^2 + (\omega_p\omega_s\Omega_m W)^2}{(1+p_0^2\Omega_m^2)^2}}{\frac{\omega_p^2}{\Omega_m^2}}\right\}$$

Having identified the transfer function for odd as well as even order elliptic filters, we can easily identify the central frequency $\omega_c \equiv \omega_0$ and the quality factor Q as

$$\omega_c^m = \frac{\sqrt{(p_0V_m)^2 + (\omega_p\omega_s\Omega_m W)^2}}{1+p_0^2\Omega_m^2} \qquad \text{and} \qquad Q_m = \frac{1}{2}\sqrt{1 + \left(\frac{\omega_p\omega_s\Omega_m W}{p_0V_m}\right)^2}$$

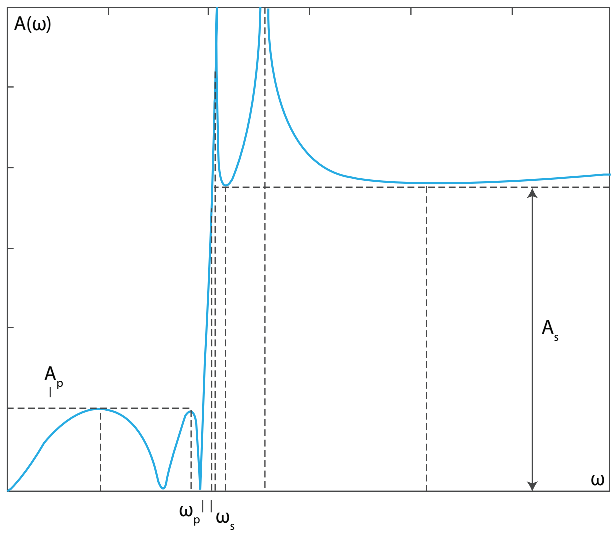

Figure 6: Loss characteristics for fifth order lowpass filter.

2.5 The loss characteristics

The loss characteristics of the low pass elliptic filter has the form

$$\begin{aligned} A(\omega) &= 10\log\left[1+\varepsilon_p^2 R_N^2(s/j,\omega_s,\omega_p,\varepsilon_s,\varepsilon_p)\right] \\ &= 10\log\left\{1+\varepsilon_p^2 \text{sn}^2\left[\kappa \text{sn}^{-1}\left(\frac{\omega}{\omega_p},\frac{\omega_p}{\omega_s}\right)+q\boldsymbol{K}_1,\varepsilon_p\varepsilon_s\right]\right\} \end{aligned}$$

A curve for this function and for a 5th order lowpass filter is shown in Figure 6.

2.6 Phase characteristics and time response function

At it is well known from the literature, the response of an LTI system to the elementary exponential signal $x(t) = Ae^{j\omega_0 t}$ has the form

$$y(t) = Ae^{j\omega_0 t}\left\{|\mathcal{H}(j\omega_0)|e^{j\measuredangle\mathcal{H}(j\omega_0)}\right\} = \left(A|\mathcal{H}(j\omega_0)|\right)\, exp\left\{j\,\left(\omega_0 t + \measuredangle\mathcal{H}(j\omega)\right)\right\}$$

and therefore the filter increases the phase of the input signal by an amount of $\measuredangle\mathcal{H}(j\omega_0)$, namely, by the phase of the filter transfer function estimated for the input signal frequency $\omega = \omega_0$. In the ideal case, the phase response of the filter is equal to zero, and no phase distortion is caused to the signal; however, these ideal filters are not realizable and in the real applications they are approximated by the well known analog filter approximations (Butterworth, Chebyshev I and II and elliptic or Cauer approximation). Another interesting case, is associated with the linear phase filters whose phase response has the simple form $\measuredangle\mathcal{H}(j\omega) = -\alpha\omega$ where $\alpha = d[\measuredangle\mathcal{H}(j\omega)]/d\omega$ is the constant group delay of those filters. It can be easily proven, that in this case the input signal is subjected a constant time delay for all frequencies, with no further distortion. In practice, the most efficient and commonly used filters of this type, are the Bessel - Thomson filters that are characterized by a maximally flat group delay for the frequency $\omega = 0$. The above described facts hold for a single frequency $\omega = \omega_0$, while, in the general case of an arbitrary input signal, they are applied to each one of the single frequency components that form its spectrum.

The phase response of the prototype lowpass elliptic filter for cutoff frequency $\omega_c = 1$, parameter values $A_p = 1$ dB, $A_s = 30$ dB and filter orders $N = 1$ to 10 is plotted in Figure 7. The defining equation of this function can be derived from the general frequency response equations for even and odd filter orders

$$\measuredangle\mathcal{H}(j\omega) = \sum_{k=1}^{N/2} \measuredangle\mathcal{H}_k(j\omega) = \sum_{k=1}^{N/2} \left\{ \tan^{-1}\left(\frac{\beta_{1k}\omega}{\beta_{0k}-\beta_{2k}\omega^2}\right) - \tan^{-1}\left(\frac{\alpha_{1k}\omega}{\alpha_{0k}-\alpha_{2k}\omega^2}\right)\right\}$$

$$\measuredangle\mathcal{H}(j\omega) = -\tan^{-1}\left(\frac{\alpha_{1k}\omega}{\alpha_{0k}}\right) + \sum_{k=1}^{(N-1)/2} \left\{ \tan^{-1}\left(\frac{\beta_{1k}\omega}{\beta_{0k}-\beta_{2k}\omega^2}\right) - \tan^{-1}\left(\frac{\alpha_{1k}\omega}{\alpha_{0k}-\alpha_{2k}\omega^2}\right)\right\}$$

for the parameters $\alpha_0 = p_0$, $\alpha_1 = 1$, $\alpha_2 = 0$, $\beta_0 = 1$, $\beta_1 = \beta_2 = 0$, $\alpha_{2k} = \beta_{2k} = 1$, $\beta_{1k} = 0$ and

$$\beta_{0k} = \frac{\omega_p^2}{\Omega_k^2}, \quad \alpha_{0k} = \frac{(p_0V_k)^2 + (\omega_p\omega_s\Omega_kW)^2}{(1+p_0^2\Omega_k^2)}, \quad \alpha_{1k} = \frac{2p_0V_k}{1+p_0^2\Omega_k^2}$$

leading to the results

$$\measuredangle\mathcal{H}(j\omega) = -\sum_{k=1}^{N/2} \frac{2p_0V_k(1+p_0^2\Omega_k^2)\omega}{(p_0V_k)^2 + (\omega_p\omega_s\Omega_kW)^2 - (1+p_0^2\Omega_k^2)\omega^2} \quad \text{(N even)}$$

$$\measuredangle\mathcal{H}(j\omega) = -\tan^{-1}\left(\frac{\omega}{p_0}\right) - \sum_{k=1}^{(N-1)/2} \frac{2p_0V_k(1+p_0^2\Omega_k^2)\omega}{(p_0V_k)^2 + (\omega_p\omega_s\Omega_kW)^2 - (1+p_0^2\Omega_k^2)\omega^2} \quad \text{(N odd)}$$

The set of parameters α_i and β_i appearing in the defining equations, are the coefficients of the polynomials of the denominator and the numerator respectively of the rational filter transfer fraction

$$\mathcal{H}(s) = \frac{\beta_0 + \beta_1 s + \beta_2 s^2 + \cdots + \beta_M s^M}{\alpha_0 + \alpha_1 s + \alpha_2 s^2 + \cdots + \alpha_N s^N}$$

The differentiation of the above equations with respect to ω allows us to derive the group delay as the negative slope of the phase response. For the case of the elliptic filters this delay for even and odd filter orders is expressed as

$$\tau(\omega) = \frac{\sum_{k=1}^{N/2} \left\{ \frac{2p_0V_k}{1+p_0^2\Omega_k^2}\left[\frac{(p_0V_k)^2 + (\omega_p\omega_s\Omega_kW)^2}{(1+p_0^2\Omega_k^2)^2} + \omega^2\right]\right\}}{\left\{\left[\frac{(p_0V_k)^2 + (\omega_p\omega_s\Omega_kW)^2}{(1+p_0^2\Omega_k^2)^2} - \omega^2\right]^2\right\}} \quad \text{(N even)}$$

$$\tau(\omega) = \frac{\sum_{k=1}^{(N-1)/2} \left\{ \frac{2p_0V_k}{1+p_0^2\Omega_k^2}\left[\frac{(p_0V_k)^2 + (\omega_p\omega_s\Omega_kW)^2}{(1+p_0^2\Omega_k^2)^2} + \omega^2\right]\right\}}{\left\{\left[\frac{(p_0V_k)^2 + (\omega_p\omega_s\Omega_kW)^2}{(1+p_0^2\Omega_k^2)^2} - \omega^2\right]^2\right\} + \frac{p_0}{p_0^2+\omega^2}} \quad \text{(N odd)}$$

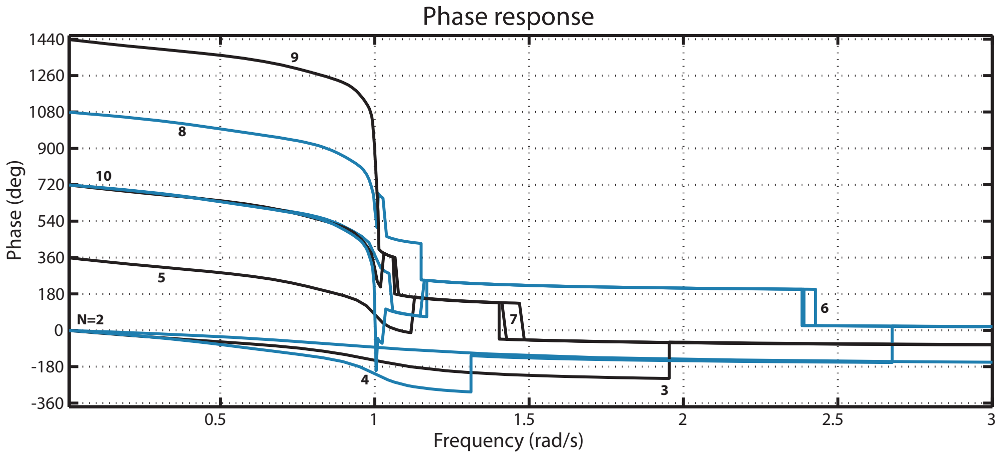

Figure 7: The phase response of the prototype lowpass elliptic filter for cutoff frequency $\omega_c = 1$, parameter values $A_p = 1$ dB, $A_s = 30$ dB and filter orders $N = 1 - 10$.

The impulse and the step response of typical examples of elliptic filters are ploted in Figure 8 and their equations have the well known exponential form characterizes the other ideal filter approximations.

3 The Design Procedure of Elliptic Filters

After the analytical description of the fundamental properties of the elliptic filters, let us now summarize the filter design procedure for the low pass prototype elliptic filter. This procedure is composed of the following steps:

1. The filter design parameters are identified, namely, the maximum passband loss A_p, the minimum stopband loss A_s, the passband edge ω_p and the stopband edge ω_s.
2. The elliptic modulus

$$k_1 = \varepsilon_p \varepsilon_s = \sqrt{\frac{10^{A_p/10} - 1}{10^{A_s/10} - 1}} = \frac{1}{\sqrt{D}} \qquad \text{where} \qquad D = \frac{10^{A_s/10} - 1}{10^{A_p/10} - 1}$$

   is estimated as well as the modulus $k_2 = \omega_p/\omega_s$ that gives the value of the filter selectivity.
3. The parameter values $k_1' = \sqrt{1 - k_1^2}$ and $k_2' = \sqrt{1 - k_2^2}$ as well as the elliptic integrals $\boldsymbol{K}_1$, $\boldsymbol{K}_1'$, $\boldsymbol{K}_2$ and $\boldsymbol{K}_2'$ are estimated using the arithmetic - geometric mean.
4. The minimum filter order is estimated as

$$N = \frac{\log(16D)}{\log(1/q(k_2))}$$

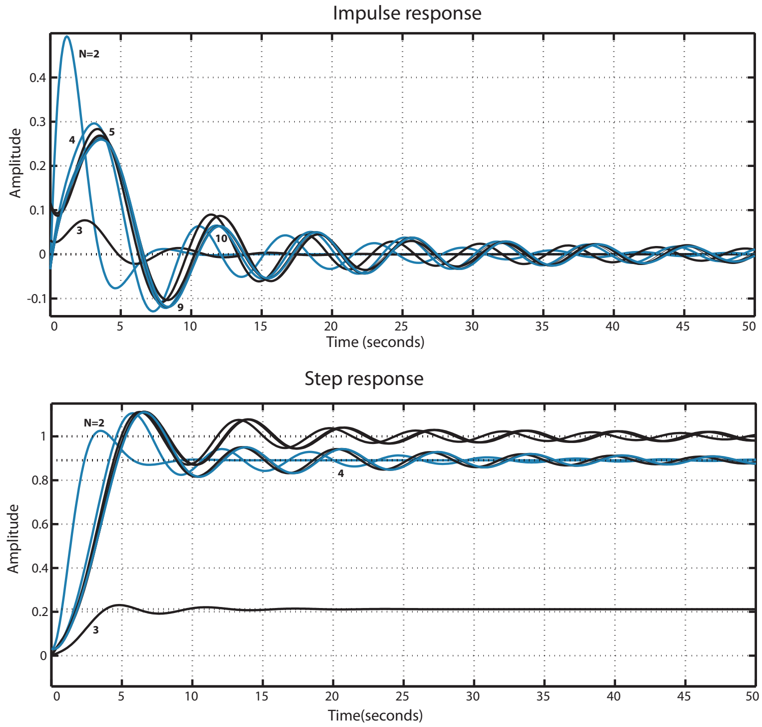

Figure 8: The impulse response and the step response of the prototype elliptic filter for cutoff frequency $\omega_c = 1$, parameter values $A_p = 1$ dB, $A_s = 30$ dB and filter orders $N = 1 - 10$.

or alternatively by the relation

$$N = \frac{\boldsymbol{K}_1'}{\boldsymbol{K}_1} \frac{\boldsymbol{K}_2}{\boldsymbol{K}_2'}$$

and rounded to the next integer number.

5. The parameters

$$p_0 = -\sqrt{\omega_s \omega_p} \frac{2\sqrt[4]{q(k_2)} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}(k_2) \sinh \left[(2n+1) \frac{\boldsymbol{K}_2}{N} \ln \frac{10^{A_p/20}+1}{10^{A_p/20}-1} \right]}{1 + 2 \sum_{n=1}^{\infty} (-1)^n q^{n^2}(k_2) \cosh \left[2n \frac{\boldsymbol{K}_2}{N} \ln \frac{10^{A_p/20}+1}{10^{A_p/20}-1} \right]}$$

$$W = \sqrt{\left(1 + \frac{p_0^2}{\omega_p^2}\right)\left(1 + \frac{p_0^2}{\omega_s^2}\right)}$$

and

$$\Omega_m = \frac{1}{\sqrt{\omega_s \omega_p}} \frac{2\sqrt[4]{q(k_2)} \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)}(k_2) \sin\left[(2n+1)\frac{\pi\mu}{N}\right]}{1 + 2\sum_{n=1}^{\infty} (-1)^n q^{n^2}(k_2) \cos\left(\frac{2n\pi\mu}{N}\right)}$$

are estimated for the values

$$\mu = \begin{cases} m & N \text{ odd} \\ m - (1/2) & N \text{ even} \end{cases}$$

where

$$m = \begin{cases} 1, 2, \ldots, (N-1)/2 & N \text{ odd} \\ 1, 2, \ldots, (N/2) & N \text{ even} \end{cases}$$

leading to the estimation of the parameters

$$\begin{aligned} V_m &= \sqrt{(1 - \omega_p^2 \Omega_m^2)(1 - \omega_s^2 \Omega_m^2)} \\ \alpha_m &= \frac{\omega_p^2}{\Omega_m^2} \\ \beta_m &= \frac{2\sigma_0 V_m}{1 + \sigma_0^2 \Omega_m^2} \\ \gamma_m &= \frac{(p_0 V_m)^2 + (\Omega_m W)^2}{(1 + p_0^2 \Omega_m^2)^2} \end{aligned}$$

6. In the last step, the filter transfer function is estimated as

$$\mathcal{H}(s) = \frac{H_0}{D(s)} \prod_{m=1}^{L} \frac{s^2 + \alpha_m}{s^2 + \beta_m s + \gamma_m}$$

where

$$\begin{aligned} L &= \begin{cases} (N-1)/2 & N \text{ odd} \\ N/2 & N \text{ even} \end{cases} \\ D(s) &= \begin{cases} s + p_0 & N \text{ odd} \\ 1 & N \text{ even} \end{cases} \end{aligned}$$

and

$$H_0 = \begin{cases} p_0 \prod_{m=1}^{L} \frac{\gamma_m}{\alpha_m} & N \text{ odd} \\ 10^{-A_p/20} \prod_{m=1}^{L} \frac{\gamma_m}{\alpha_m} & N \text{ even} \end{cases}$$

and the amplitude and phase response are derived and plotted.

The application of the above procedure for the design of a filter with prescribed specifications is shown in the next example.

3.1 A low pass elliptic filter design example

To illustrate the above described elliptic filter design procedure, let us design an elliptic filter with selectivity $k_2 = \omega_p/\omega_s = 0.95$ and attenuation levels $A_p = 0.3$ dB and $A_s = 60$ dB.

We start with the identification of the minimum filter order via the equation

$$N = \log(16D)/\log[1/q(k_2)]$$

The value of the D parameter is estimated as

$$D = \frac{10^{A_s}/10 - 1}{10^{A_p/10} - 1} = \frac{10^{60/10} - 1}{10^{0.3/10} - 1} = \frac{999999}{0.071519} = 13982284.427914$$

On the other hand, to estimate the function $q(k_2) = q(0.95)$ we can apply two alternative techniques, and since this review is actually a tutorial on elliptic filters, let us present both of them. The first technique uses the defining equation of this function, namely the equation $q(k_2) = \exp(-\pi \boldsymbol{K}_2'/\boldsymbol{K_2})$. To estimate the elliptic integrals we can use the arithmetic geometric mean, a computationally simple and very efficient method that provides the results

$$\boldsymbol{K}_2 = \boldsymbol{K}(k_2) = \boldsymbol{K}(0.95) = 2.590011$$

and

$$\boldsymbol{K}_2' = \boldsymbol{K}(k_2') = \boldsymbol{K}(\sqrt{1 - k_2^2}) = \boldsymbol{K}(0.312249) = 1.611337$$

Therefore,

$$\begin{aligned} q(k_2) = q(0.95) \; &= \; \exp\left(-\pi\frac{\boldsymbol{K}_2'}{\boldsymbol{K}_2}\right) = \exp\left(-3.14159 \times \frac{1.611337}{2.590011}\right) \\ &= \; \exp(-1.954493) = 0.141636 \end{aligned}$$

The second method is the prior estimation of the auxiliary parameter

$$q_0 = \frac{1}{2}\left(\frac{1-\sqrt{k_2'}}{1+\sqrt{k_2'}}\right) = \frac{1}{2}\left(\frac{1-\sqrt{0.312249}}{1+\sqrt{0.312249}}\right) = \frac{1}{2}\left(\frac{0.441207}{1.558972}\right) = 0.141522$$

and then the computation of the required value via the approximation

$$q(k_2) = q(0.95) = q_0 + 2q_0^5 + 15q_0^9 = 0.141522 + 2(0.141522)^5 + 15(0.141522)^9 = 0.141636$$

Even though both methods reach the same result, the second one is clearly more preferable, since it does not use elliptic integrals. Therefore, the minimal filter order is equal to

$$N = \frac{\log(16D)}{\log[1/q(k_2)]} = \frac{\log(16 \times 13982284.427914)}{\log(1/0.141636)} = \frac{8.349698}{0.848826} = 9.836756$$

and since this value is a decimal number (as happens in almost cases) it has to be rounded to the next available integer leading thus to the value $N = 10$.

Since the prescribed specifications do not include the frequencies ω_p and ω_s but only the selectivity factor $k_2 = 0.95$, we can set $\omega_p = \sqrt{k_2} = \sqrt{0.95} = 0.974679$ and furthermore

$\omega_s = 1/\sqrt{k_2} = 1/\sqrt{0.95} = 1.025978$ to construct a normalized filter with a cutoff frequency $\omega_c = \sqrt{\omega_p \omega_s} = 1$ (this is a commonly used approach). Therefore,

$$j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1} = j\frac{2.590011}{10\times 3.14159}\ln\frac{10^{0.3/20}+1}{10^{0.3/20}-1} = 0.334365j$$

and the parameter p_0 is estimated as

$$p_0 = j\omega_p \text{sn}\left[j\frac{\boldsymbol{K}_2}{N\pi}\ln\frac{10^{A_p/20}+1}{10^{A_p/20}-1}, \frac{\omega_p}{\omega_s}\right] = j\,0.974679\,\text{sn}(j\,0.334625, 0.75)$$

At this point we have to estimate the Jacobian elliptic sinus function, and this is not a trivial procedure. One way to do it, is to develop a user defined software function to compute this value from its series expansion in terms of the θ function. However, since this is not a crucial task in this presentation, we simply use the [SN,CN,DN]=ellipj(u,k) MATLAB function [12][13] that gets as inputs the parameters u and k and returns the values of the elliptic functions sn, cn and du. The problem is that this MATLAB routine has been designed to work with real arguments, but in this situation we want to estimate the elliptic sinus of an imaginary quantity. To overcome this limitation we can use the identity

$$\text{sn}(ju, m) = j\text{sc}(u, m_1) = j\frac{\text{sn}(u, m_1)}{\text{cn}(u, m_1)}$$

where $u = 0.334625$, $m = k_2^2 = (0.95)^2 = 0.9025$ and

$$m_1 = (k_2')^2 = 1 - k_2^2 = 1 - m_1 = 1 - 0.9025 = 0.0975$$

Therefore we have $\text{sn}(0.334625, 0.0975) = 0.327852$ and $\text{cn}(0.334625, 0.0975) = 0.944728$, reaching the result

$$\text{sn}(j\,0.334625, 0.0975) = j\frac{0.327852}{0.944728} = j\,0.347033$$

which in turn gives

$$p_0 = j\omega_p(j\,0.347033) = j\,(0.974679)(j\,0.347033) = -0.338245$$

In the final step we use this p_0 value to estimate the W parameter as

$$W = \sqrt{\left(1+\frac{p_0^2}{\omega_p^2}\right)\left(1+\frac{p_0^2}{\omega_s^2}\right)} = \sqrt{\left(1+\frac{(-0.338245)^2}{(0.974679)^2}\right)\left(1+\frac{(-0.338245)^2}{(1.025978)^2}\right)} = 1.114634$$

At this stage of the design procedure, we have in our disposal all of the required information for the estimation of the filter transfer function and lets start with the identification of its zero values. The Ω_m parameters are estimated as

$$\Omega_m = \frac{1}{\omega_s}\text{sn}\left[\frac{2m-1}{N}\boldsymbol{K}_2, \frac{\omega_p}{\omega_s}\right] = 0.974679\text{sn}[0.259001(2m-1), 0.75], \quad m = 1,2,3,4,5$$

Using the ellipj MATLAB function we get

$$\Omega_1 = 0.247614,\ \ \Omega_2 = 0.648846,\ \ \Omega_3 = 0.868917,\ \ \Omega_4 = 0.959913,\ \ \Omega_5 = 0.970943$$

and therefore, the zeros of the transfer function are the following:

$$z_{1,2} = \pm\frac{j}{\Omega_1} = \pm\frac{j}{0.247614} = \pm 4.038539j \quad z_{3,4} = \pm\frac{j}{\Omega_2} = \pm\frac{j}{0.648846} = \pm 1.545961j$$
$$z_{5,6} = \pm\frac{j}{\Omega_3} = \pm\frac{j}{0.868917} = \pm 1.150857j \quad z_{7,8} = \pm\frac{j}{\Omega_4} = \pm\frac{j}{0.959913} = \pm 1.041760j$$
$$z_{9,10} \pm \frac{j}{\Omega_5} = \pm\frac{j}{0.970943} = \pm 1.029926j$$

On the other hand, to identify the poles of the transfer function, we need the values

$$V_m = \sqrt{(1-\omega_p^2\Omega_m^2)(1-\omega_s^2\Omega_m^2)}$$

(for $m = 1, 2, 3, 4, 5$) that are estimated as

$$V_1 = 0.938601,\ V_2 = 0.578042,\ V_3 = 0.240895,\ V_4 = 0.061221,\ V_5 = 0.028267$$

leading, in turn, to the results

$$\begin{aligned} p_{1,2} &= -0.315265 \pm 0.275998j \\ p_{3,4} &= -0.186534 \pm 0.689991j \\ p_{5,6} &= -0.075002 \pm 0.891514j \\ p_{7,8} &= -0.018732 \pm 0.967913j \\ p_{9,10} &= -0.008630 \pm 0.976882j \end{aligned}$$

The normalization constant H_0 can be identified from its defining equation for even filter order and it is found equal to $H_0 = 0.001197$.

To construct the filter transfer function we have to build the contribution of each pole and each zero and the results are the following:

- Zeros $z_{1,2}$ contribute to the numerator the term $s^2 + 16.309797$
- Zeros $z_{3,4}$ contribute to the numerator the term $s^2 + 2.389995$
- Zeros $z_{5,6}$ contribute to the numerator the term $s^2 + 1.324471$
- Zeros $z_{7,8}$ contribute to the numerator the term $s^2 + 1.085263$
- Zeros $z_{9,10}$ contribute to the numerator the term $s^2 + 1.060747$
- Poles $p_{1,2}$ contribute to the denominator the term $s^2 + 0.630539s + 0.175566$
- Poles $p_{3,4}$ contribute to the denominator the term $s^2 + 0.373068s + 0.510881$
- Poles $p_{5,6}$ contribute to the denominator the term $s^2 + 0.150004s + 0.800422$
- Poles $p_{7,8}$ contribute to the denominator the term $s^2 + 0.037464s + 0.937205$
- Poles $p_{9,10}$ contribute to the denominator the term $s^2 + 0.017260s + 0.954372$

After the identification of all those terms, we can construct the polynomial of the numerator by multiplying the first five terms associated with the five zeros, while, the denominator polynomial can be constructed in a similar way by multiplying the five terms associated with the five poles. The degree of both polynomials is equal to $N = 10$. The numerator polynomial contains only even powers of s, while the denominator polynomial contains all the powers of s between 0 and 10. To complete the design, the numerator polynomial must

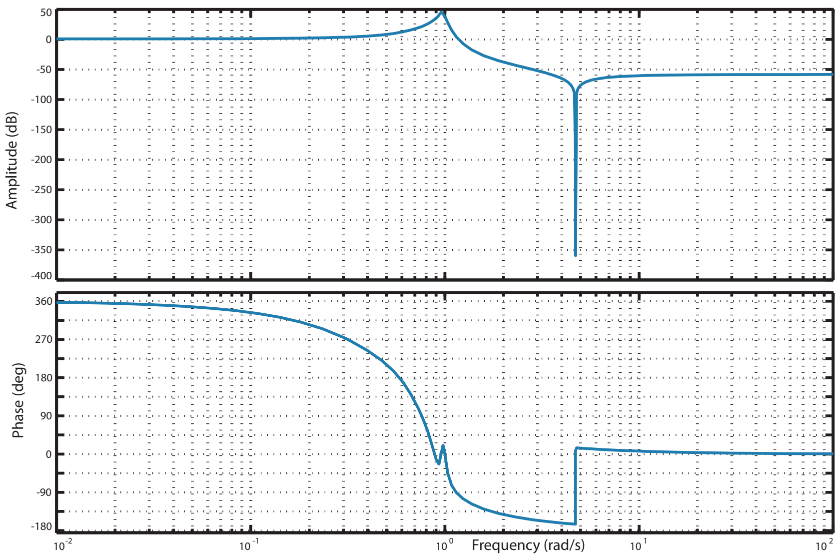

Figure 9: The amplitude and the phase response of the designed example filter.

be multiplied by the normalization constant H_0. The final result expressed with an accuracy of three decimal digits has the form

$$\mathcal{H}(s) = \frac{0.001197(s^{10}+22.170s^8+1.078s^6+2.144s^4+1.841s^2+59.433)}{s^{10}+1.20s^9+3.82s^8+3.66s^7+5.45s^6+4.01s^5+3.50s^4+1.85s^3+0.94s^2+0.29s+0.064}$$

The amplitude and phase response of this filter are plotted in Figure 9. However, it is clear that these results are only approximative, since we used the approximated expressions. If we construct the same filter using the appropriate MATLAB functions the results are

$$\begin{aligned} z_{1,2} &= \pm 4.076817j & p_{1,2} &= -0.327805 \pm 0.285514j \\ z_{3,4} &= \pm 1.583155j & p_{3,4} &= -0.195119 \pm 0.705739j \\ z_{5,6} &= \pm 1.195265j & p_{5,6} &= -0.085341 \pm 0.902603j \\ z_{7,8} &= \pm 1.084517j & p_{7,8} &= -0.031622 \pm 0.977975j \\ z_{9,10} &= \pm 1.051635j & p_{9,10} &= -0.007805 \pm 1.002470j \end{aligned}$$

and $H_0 = 9.9999 \times 10^{-4}$, and as it can be easily seen, our solution is associated with a good precision. In practice and in real applications, the elliptic filters are designed using specialized software packages due to the complications associated with their mathematical foundations.

4 Conclusions

The objective of this review paper was the detailed and rigorous mathematical presentation of the main aspects, properties and features of the low pass analog elliptic filter approximation.

These features include the minimum filter order that meets the prescribed specifications, the poles and zeros of the Chebyshev rational function, the filter transfer function, the amplitude and the phase response, the group delay, as well as the impulse and step responses. An appendix describing the main features and properties of the elliptic and theta functions is also included for the sake of completeness.

Competing Interests

The author declares that no competing interests exist.

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Appendix - Elliptic Integrals and Related Functions

Since the elliptic functions and the related stuff are not very common and they are rarely used in practice, this appendix introduces the interested reader to the most important aspects and properties of them.

1. Elliptic Integrals

Elliptic integrals were introduced by Wallis and Newton and they were used in a systematic way by Legendre for the study of classical mechanical problems such as the single pendulum. They are considered as generalizations of the inverse trigonometric functions, and they are defined as

$$f(x) = \int \frac{\alpha(x) + \beta(x)}{\gamma(x) + \delta(x)\zeta(x)}$$

In the above equation, the right-hand functions are polynomial of x, with the highest power in $\zeta(x)$ to have a value of 3 or 4. These functions depend on two variables, one of them is associated with the upper limit of integration. There are three kinds of elliptic integrals, namely, first, second, and third kind, each one of them is characterized as complete or incomplete.

The incomplete integral of first kind is defined as

$$\boldsymbol{F}(\varphi, k) = \int_0^{\sin\varphi} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$$

where $0 \leq k^2 \leq 1$ and $0 \leq \varphi \leq 1$. The k parameter is known as elliptic module, and it can be defined alternatively in terms of the parameter $m = k^2$ or the angle $\alpha = \sin^{-1} k$. Performing the change of variable $t = \sin\vartheta$ we have that $dt = \cos\vartheta d\vartheta = \sqrt{1-t^2}\, d\vartheta$ and the integral gets the form

$$\boldsymbol{F}(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}$$

where $0 \leq \varphi \leq \pi/2$. Regarding the complete elliptic integral of first kind it is associated with the upper integration limit value $\varphi = \pi/2$ and therefore we have

$$\boldsymbol{F}(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}$$

On the other hand, the incomplete and complete elliptic integral of second kind are

$$\boldsymbol{E}(\varphi, k) = \int_0^{\sin\varphi} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}dt} = \int_0^{\varphi} \sqrt{1-k^2\sin^2\vartheta}$$

and

$$\boldsymbol{E}(k) = \int_0^{1} \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}dt} = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\vartheta}$$

while, the corresponding expressions for the incomplete and complete elliptic integrals of third kind have the form

$$\begin{aligned}
\boldsymbol{\Pi}(\varphi, \eta, k) &= \int_0^{\sin\varphi} \frac{dt}{(1+\eta t^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\varphi} \frac{d\vartheta}{(1+\eta\sin^2\vartheta)(1-k^2\sin^2\vartheta)} \\
\boldsymbol{\Pi}(\eta, k) &= \int_0^{1} \frac{dt}{(1+\eta t^2)\sqrt{(1-t^2)(1-k^2t^2)}} = \int_0^{\pi/2} \frac{d\vartheta}{(1+\eta\sin^2\vartheta)(1-k^2\sin^2\vartheta)}
\end{aligned}$$

In the literature, the complete elliptic integrals of first and second kind are denoted as $\boldsymbol{K}$ and $\boldsymbol{E}$, respectively, and therefore we have,

$$\boldsymbol{F}\left(\varphi = \frac{\pi}{2}, k\right) = \boldsymbol{F}(\sin\varphi = 1, k) = \boldsymbol{K}(K) = \boldsymbol{K}$$
$$\boldsymbol{E}\left(\varphi = \frac{\pi}{2}, k\right) = \boldsymbol{E}(\sin\varphi = 1, k) = \boldsymbol{E}(K) = \boldsymbol{E}$$

Defining the parameter $k' = \sqrt{1-k^2}$ and performing the substitution $\upsilon = \tan\vartheta$ we define the so-called complementary form for these integrals

$$\boldsymbol{F}\left(\varphi = \frac{\pi}{2}, k'\right) = \boldsymbol{F}(\sin\varphi = 1, k') = \boldsymbol{K}(K') = \boldsymbol{K}'$$
$$\boldsymbol{E}\left(\varphi = \frac{\pi}{2}, k'\right) = \boldsymbol{E}(\sin\varphi = 1, k') = \boldsymbol{E}(K') = \boldsymbol{E}'$$

2. Elliptic Functions

The elliptic functions are closely related to the elliptic integrals and in fact they are associated with the inversion of those integrals. In this appendix the focus is given to a specific class of elliptic functions, namely the Jacobian elliptic functions, since they are used for the design of elliptic filters. These functions are defined via the inversion of the incomplete elliptic integral of first kind. Expressing this integral in the form

$$u = F(\varphi, k) = \int_0^{\varphi} \frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}}$$

the upper limit of integration is known as Jacobian amplitude and it is denoted as $\text{amp}(u,k)$. Therefore, we have $\varphi(u,k) = F^{-1}(u,k) = \text{amp}(u,k)$. Based on this fact, we can define the three most well known elliptic functions as

$$\begin{aligned}
\text{sn}(u,k) &= \sin\varphi(u,k) = \sin[\text{amp}(u,k)] \\
\text{cn}(u,k) &= \cos\varphi(u,k) = \cos[\text{amp}(u,k)] \\
\text{dn}(u,k) &= \sqrt{1-k^2\sin^2\varphi(u,k)} = \sqrt{1-k^2\sin^2[\text{amp}(u,k)]}
\end{aligned}$$

These functions are generalizations of the well known trigonometric and hyperbolic functions for the parameter values $k=0$ and $k=1$, since, based on their defining equations we can easily see that $\text{sn}(u,0) = \sin u$, $\text{cn}(u,0) = \cos u$, $\text{dn}(u,0) = 1$, $\text{cd}(u,0) = \cos u$, and furthermore, $\text{sn}(u,1) = \tanh u$, $\text{cn}(u,1) = \text{sech}u$, $\text{dn}(u,1) = \text{sech}u$, $\text{cd}(u,1) = 1$. Furthermore, using the auxiliary parameter $k' = \sqrt{1-k^2}$, it can be easily proven via their defining equations, that

$$\begin{aligned}
&\text{sn}^2(u,k) + \text{cn}^2(u,k) = 1, \quad k^2\text{sn}^2(u,k) + \text{dn}^2(u,k) = 1 \\
&\text{dn}^2(u,k) - k^2\text{cn}^2(u,k) = k'^2 \\
&k'^2\text{sn}^2(u,k) + \text{cn}^2(u,k) = \text{dn}^2(u,k) \\
&\text{sn}^2(u,k) = \frac{1-\text{cd}^2(u,k)}{1-k^2\text{cd}^2(u,k)}
\end{aligned}$$

The elliptic functions used in the design of the elliptic filters, are the even function $\text{sn}(u,k)$ and the odd function $\text{cd}(u,k)$. These functions are related via the expressions

$$\text{cd}(u,k) = \text{sn}(u+\boldsymbol{K},k) = \text{sn}(\boldsymbol{K}-u,k)$$

and as it can be proven they also satisfy the relations

$$\begin{aligned}
&\mathsf{cd}\,[u+(2m-1)\boldsymbol{K},k] = (-1)^m \mathsf{sn}(u,k) \quad (\forall m \in N) \\
&\mathsf{cd}\,[u+2m\boldsymbol{K},k] = (-1)^m \mathsf{cd}(u,k) \quad (\forall m \in N) \\
&\mathsf{cd}\,[u+j\boldsymbol{K}',k] = 1/[k\mathsf{cd}(u,k)] \quad (\forall u \in N) \\
&\mathsf{cd}\,[ju,k] = 1/\mathsf{dn}(u,k') \quad (\forall u \in N)
\end{aligned}$$

The Jacobian elliptic function are periodic with respect to the real as well as to the imaginary axis of the complex plane with period values for these two directions $T_x = 4\boldsymbol{K}$ and $T_y = 2j\boldsymbol{K}'$, respectively. More specifically, for the functions $\mathsf{sn}(u,k)$ and $\mathsf{cn}(u,k)$ and for each value of u we have

$$\mathsf{sn}(u+4m\boldsymbol{K},k) = \mathsf{sn}(u,k) \quad \text{and} \quad \mathsf{cn}(u+2jm\boldsymbol{K}',k) = \mathsf{cn}(u,k)$$

where m and integer. It is proven that these functions also define a conformal mapping from the u to the ω plane, with the smallest area of the u plane that is mapped to the whole ω plane to define the so-called fundamental area.

3. The Nome and ϑ Functions

The nome Jacobian function q as well as its complementary function q_c are defined in terms of the complete elliptic integrals $\boldsymbol{K}$ and $\boldsymbol{K}'$ as

$$\begin{aligned}
q &= q(k) = \exp\left(-\pi\frac{\boldsymbol{K}'}{\boldsymbol{K}}\right) = \exp\left(-\pi\frac{\boldsymbol{K}(\sqrt{1-k^2})}{\boldsymbol{K}(k)}\right) \\
q_c &= q_c(k) = \exp\left(-\pi\frac{\boldsymbol{K}}{\boldsymbol{K}'}\right) = \exp\left(-\pi\frac{\boldsymbol{K}(k)}{\boldsymbol{K}(\sqrt{1-k^2})}\right)
\end{aligned}$$

It can be easily proven that these two functions satisfy the equation $\ln(q)\times\ln(q_c) = \pi^2$. The estimation of the $q(k)$ is based to the definition of the variable $k = \sin\alpha$ as well as the auxiliary parameter

$$\varepsilon = \varepsilon(k) = \frac{1}{2}\left(\frac{1-\sqrt{\cos\alpha}}{1+\sqrt{\cos\alpha}}\right) = \frac{1}{2}\left(\frac{1-\sqrt[4]{1-k^2}}{1+\sqrt[4]{1-k^2}}\right)$$

In this case it can be proven that the function $q(k)$ can be expressed as a series in the form

$$q(k) = \varepsilon(k) + 2\varepsilon^5(k) + 15\varepsilon^9(k) + 150\varepsilon^{13}(k) + 1707\varepsilon^{17}(k)$$

This series is an excellent approximation of the function $q(k)$ and it converges for the interval $0 \le \alpha \le \pi/4$ or $0 \le k \le 1/\sqrt{2}$. On the other hand, the complementary function $q_c(k)$ can be approximated as the value of the series

$$q_c(k) = \varepsilon_c(k) + 2\varepsilon_c^5(k) + 15\varepsilon_c^9(k) + 150\varepsilon_c^{13}(k) + 1707_c\varepsilon^{17}(k)$$

that converges in the interval $\pi/4 \le \alpha \le \pi/2$ or $1/\sqrt{2} \le k \le 1$ where

$$\varepsilon_c = \varepsilon_c(k) = \frac{1}{2}\left(\frac{1-\sqrt{\sin\alpha}}{1+\sqrt{\sin\alpha}}\right) = \frac{1}{2}\left(\frac{1-\sqrt(k)}{1+\sqrt{k}}\right)$$

The nome Jacobian function allows the definition and estimation of the functions $\vartheta_i(u,q)$ $(i = 1,2,3,4)$

as

$$\begin{aligned}
\vartheta_1(u,q) &= 2\sqrt[4]{q}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)}\sin[(2n+1)\pi u] \\
\vartheta_2(u,q) &= 2\sqrt[4]{q}\sum_{n=0}^{\infty} q^{n(n+1)}\cos[(2n+1)\pi u] \\
\vartheta_3(u,q) &= 1+2\sum_{n=1}^{\infty} q^{n^2}\cos(2n\pi u) \\
\vartheta_4(u,q) &= 1+2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos(2n\pi u)
\end{aligned}$$

The ϑ functions are considered useful tools in the study and use of elliptic functions. It can be proven that each ellpitic function can be expressed as the ratio of two ϑ functions. The mathematical expressions that describe this type of relationships are based to the complete elliptic integral of first kind, expressed as a function of the arithmetic - geometric mean $M(x,y)$ which is defined as the common limit of convergence of the sequences $\alpha_n = (\alpha_{n-1}+\beta_{n-1})/2$ and $\beta_n = \sqrt{\alpha_{n-1}\beta_{n-1}}$ with $\alpha_0 = x$ and $\beta_0 = y$. In this case it can be proven that the complete elliptic integral of first kind can be estimated as

$$\boldsymbol{k}(k) = \int_0^{\pi/2}\frac{d\vartheta}{\sqrt{1-k^2\sin^2\vartheta}} = \frac{\pi}{2M(1,\sqrt{1-k^2})} = \frac{\pi}{M(1-k,1+k)}$$

Based on this result, the Jacobian elliptic sinus function can be estimated as the ratio

$$\text{sn}(u,k) = \frac{1}{\sqrt{k}}\frac{\vartheta_1\left[\dfrac{u}{2\boldsymbol{K}(k)},q(k)\right]}{\vartheta_4\left[\dfrac{u}{2\boldsymbol{K}(k)},q(k)\right]} = \frac{1}{\sqrt{k}}\frac{\vartheta_1\left[u\dfrac{M(1,\sqrt{1-k^2})}{\pi},\exp\left(-\pi\dfrac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]}{\vartheta_4\left[u\dfrac{M(1,\sqrt{1-k^2})}{\pi},\exp\left(-\pi\dfrac{M(1,\sqrt{1-k^2})}{M(1,k)}\right)\right]}$$

Similar expressions can be constructed for the remaining Jacobian elliptic functions.

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